Some Aspects of Happel's Theorem and the Auslander-Reiten Triangles

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MTH 619, Representation Theory of Quivers
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This Land is ~~my~~ Land: Part I

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We have seen $D^b(mod - A)$ in some previous presentations. It is a Hom-finite, Krull-Schmidt category when A is finite dimensional. They are crucial in understanding some deeper homological information. However, (bounded) derived category $D^b(mod - A)$ is quite an abstract setting where we are concerned with a class of complexes in $K^b(mod - A)$ modulo quasi-isomorphism.

Triangles of our interests, namely Auslander-Reiten triangles

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It would be nice to find some playground where we can extract AR triangles in $D^b(mod - A)$ without worrying about this derived category. For this we turn to the triangulated structures in $D^b(mod - A)$ where triangles are of form

$$X \xrightarrow{f} Y \to Cone(f) \to X[1] \tag{2}$$

but not easy to see.

Another triangulated category would be useful to us. We will look at the stable category of modules over a repetitive algebra $\hat{A}$.

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Theorem (Happel)

Stable category of a Frobenius category is a triangulated category.

(I will try to convince this in my next part of the talk.) Let us say that $mod - \hat{A}$ are modules over a repetitive algebra $\hat{A}$ and the stable category (where we factor out all the morphism which factor through a projective) of $mod - \hat{A}$ will admit almost split triangles.

Happel describes a functor

$$\mu : D^b(mod - A) \to \underline{mod} - \hat{A} \tag{3}$$

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Our goal is to understand the AR triangles in $D^b(mod - A)$ using the familiarity of triangles of modules in $\underline{mod} - \hat{A}$. However, when $gldim(A)$ is infinite, then it is interesting to find the differences between $D^b(mod - A)$ and $\underline{mod} - \hat{A}$. For such example, see [Happel, Keller, Reiten, 2007] for Gorenstein algebra A.

In this talk, we wish to discuss the following theorem(s)

Theorem (Happel, 1991)

Let A be a finite-dimensional k-algebra. Then the bounded derived category $D^b(mod - A)$ admits Auslander-Reiten triangles if and only if A has finite global dimension (gldim $< \infty$).

And show a triangulated category $\underline{mod} - \hat{A}$ which is equivalent to $D^b(mod - A)$ which helps us to study AR triangles.

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Definition

Let $\mathcal{C}$ be an additive category, then define Σ to be automorphism of $\mathcal{C}$. The automorphism $\Sigma : \mathcal{C} \to \mathcal{C}$ is called the translation functor of $\mathcal{C}$.

Define a sextuple (X, Y, Z, u, v, w) given by the objects $(X, Y, Z) \in \mathcal{C}$ and morphisms $u : X \to Y$, $v : Y \to Z$ and $w : \Sigma X$. Basically, we can write the sextuple as

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} Z \to \Sigma X \quad (4)$$

We can find the morphisms between two sextuples

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Sigma f} \\ X' & \xrightarrow[u']{} & Y' & \xrightarrow[v']{} & Z' & \xrightarrow[w']{} & \Sigma X' \end{array}$$

such that each square commutes.

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\end{array}
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such that each square commutes.

Now, consider a set T of all sextuples in $\mathcal{C}$, then set T is a triangulation of $\mathcal{C}$ if they satisfy Verdier's axioms. Any element of T is called a **triangle**.

These Verdier's axioms are as follows.

TR1 For every object X, the following is a distinguished triangle

$$X \to X \to 0 \to \Sigma X \tag{5}$$

(Also, consider any morphism $u : X \to Y$, we can get a triangle $X \xrightarrow{u} Y \to cone(u) \to \Sigma X$)
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Any triangle isomorphic to a distinguished triangle is a distinguished triangle.

TR2 The following is a distinguished triangle precisely

$$X \xrightarrow{u} Y \to Z \to \Sigma X \tag{6}$$

if

$$Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y \tag{7}$$

is a distinguished triangle.

TR3 The following diagram of morphism between two triangles commute

$$
\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Sigma f} \\
X' & \xrightarrow[u']{} & Y' & \xrightarrow[v']{} & Z' & \xrightarrow[w']{} & \Sigma X'
\end{array}
$$

TR4 Also called *octahedral axiom*. Consider three triangles and the axioms states that their mapping cones will form a distinguished triangle. [We can ignore this for this talk but they are crucial in determining the non-functoriality of mapping cones[1] in triangulated categories.]

[1] https://aayushayh.blogspot.com/2025/09/the-quill-24-failure-of-functorial.html

Definition

An additive category $\mathcal{C}$ with translation functor Σ which admits triangulation is called a **triangulated** category $(\mathcal{C}, \Sigma, T)$.

Example of our interest: Derived category $D(A)$ of an abelian category A is a triangulated category and so is $D^b(A)$.

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We study triangulated categories because it generalizes short exact sequences to distinguished triangles

$$0 \to X \to Y \to Z \to 0 \tag{8}$$

in Abelian category A is generalized as a triangle in $D^b(mod - A)$

$$X \to Y \to Z \to X[1] \tag{9}$$

Repetitive Algebra

We will introduce our next important ingredient, namely **Repetitive Algebra** $\hat{A}$ of an algebra A.

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A repetitive algebra $\hat{A}$ is a $k-$*algebra* defined for A to be an 'infinite-dimensional' algebra whose underlying vector space is given by a $\mathbb{Z}-$grading

$$\hat{A} = \left(\bigoplus_{i \in \mathbb{Z}} A_i \right) \bigoplus \left(\bigoplus_{i \in \mathbb{Z}} (DA)_i \right) \tag{10}$$

where $DA = Hom_k(A, k)$ which a bi-module A. Let (a_i, ϕ_i) be an element in $\hat{A}$, then multiplication is defined as

$$(a_i, \phi_i) \cdot (b_i, \psi_i) = (a_i b_i, a_{i+1}\psi_1 + \phi_i b_i) \tag{11}$$

Just imagine an infinite matrix algebra where the diagonal entries are A_i and the sub-diagonal entries $(i, i-1)$ are DA_i, moreover, everything else is zero.

$$\hat{A} = \begin{pmatrix} \ddots & & & & \\ & A_{i-1} & & & 0 \\ & DA_{i-1} & A_i & & \\ & & DA_i & A_{i+1} & \\ & & & \ddots & \ddots \\ & 0 & & & \end{pmatrix}$$

also $DA \cdot DA = 0$.

We can write $\hat{A}$-modules (which are also $\mathbb{Z}$-graded in same way) as $M = (M_i, f_i)$ where M_i are A-modules and f_i are A-linear maps $DA \cdot M_i \to M_{i+1}$ (which makes it actually a complex!)

$$\cdots \to M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_0} M_{i+1} \xrightarrow{f_{i+1}} \cdots$$

Algebra A	**Repetitive Algebra** $\hat{A}$
Finite-dimensional	Infinite-dimensional
Unital	Non-unital (Locally Bounded)
In general, not self-injective	Self-Injective (Frobenius)

Two Facts

1. *$\hat{A}$ is a locally bounded algebra.*
2. *$\hat{A}$ is self-injective algebra.*

A quiver corresponding to a repetitive algebra $\hat{A}$ is an infinite quiver (containing infinite vertices). However, at any vertex, it has only 'finite' number of incoming and outgoing arrows and the vector space of paths between two vertices are also finite-dimensional.

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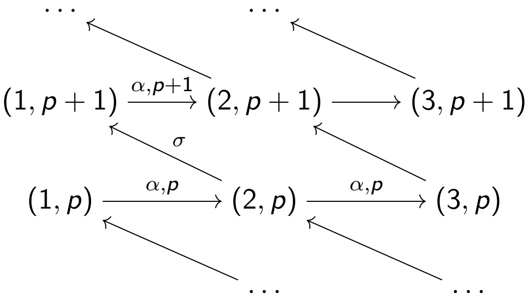
However, a module-theoretic definition is that k-algebra A is locally bounded if it admits a complete set of idempotents $\{e_i\}$ and projective module $P_i = Ae_i$ (or e_iA) has finite length (the composition series of its sub-modules exist). Simple modules are of finite length by definition $0 \subset S$.

Let us do a (quick) construction of (infinite) repetitive quiver. Say A_3, $1 \to 2 \to 3$, the quiver corresponding to $\hat{A}$ has vertices $Q_0 \times \mathbb{Z}$

$$Q_0 \times \mathbb{Z} = \{(i, p) | i \in Q_0, p \in Z\} \tag{12}$$

and edges as in following quiver



$$\alpha, p : (i, p) \to (j, p) \tag{13}$$

$$\sigma : (j, p) \to (i, p + 1) \tag{14}$$

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So, it is nice 'enough' to have a definition of finiteness locally and projective modules to have finite length. Our algebra $\hat{A}$ also has to be self-injective algebra. This is necessary for our Happel's functor.

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In fact, an algebra which is locally bounded and self-injective is called Frobenius algebra. Our repetitive algebra is a Frobenius algebra. In fact the category of modules over $\hat{A}$ (mod-$\hat{A}$) is called Frobenius category (which is a categorification of Frobenius algebra).

Following Quillen, we say that a exact category $\mathcal{C}$ possesses short exact sequences. Then the following definition, due to Happel, follows

Definition (Frobenius Category)

An exact category $\mathcal{C}$ with a class of short exact sequences S is called a Frobenius category if it has enough $S-$projectives and enough $S-$injectives and if moreover, the $S-$projectives coincide with $S-$injectives.

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Theorem (Happel,1988)

Let A be a finite-dimensional k-algebra and $\hat{A}$ be the repetitive algebra associated with A. Then mod $\hat{A}$ is a Frobenius category.

We have obtained mod $-\hat{A}$ as a Frobenius category. The next step is to connect it using a triangulated structure to $D^b(mod - A)$ which has AR triangles.

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For a triangulated category , we write AR triangles as distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{w} X[1] \tag{15}$$

where X, Z are indecomposables, f is an left almost split and g is a right almost split (f, g are irreducibles), for example

$$\tau Z \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{w} \tau Z[1] \tag{16}$$

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Studying the shapes of these Auslander-Reiten triangle are interesting in their own right. For a trivial case, $A_2, 1 \to 2$, we can directly see these AR triangles. The AR sequence in $mod - KQ$ is given by

$$0 \to S_1 \to P_1 \to S_2 \to 0 \tag{17}$$

where $S_1 = \tau S_2$. The AR triangle in $D^b(mod - A)$ corresponding to this AR sequence is

$$S_1 \to P_1 \to S_2 \to S_1[1] \tag{18}$$

but non-trivial examples are hard in $D^b(mod - A)$.

But one must be careful with dealing with the Auslander-Reiten translate in $D^b(mod - A)$ or any other triangulated category. They are not exactly same as AR translate in $mod - A$. For instance, projectives (think of them as a stalk complex) also has an AR translate $\tau = \mathbb{S} \cdot [-1]$ where $\mathbb{S}$ is Serre functor. This is not true for projectives in $mod - A$.

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One may use the program **GAP-QPA** to do programs to check the Auslander-Reiten translate in $D^b(mod - A)$ (including many more things.)

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The QPA package for GAP 4.12
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Version 1.36; April 7th, 2025
Copyright (C) 2025 The QPA-team,
                https://folk.ntnu.no/oyvinso/QPA/
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Theorem (Happel)

Stable category of a Frobenius category is a triangulated category.

Constructing stable (module) category of mod $-\hat{A}$ is enough to use the Happel's equivalence in finite global dimensions of A. Given a Frobenius category mod $-\hat{A}$, stable category $\underline{mod} - \hat{A}$ is constructed by factoring out all the morphisms which factor through a projective.

Definition (Stable Category)

A stable category $\underline{\mathcal{C}}$ is an exact category $\mathcal{C}$ defined as a quotient category where the objects are same as $\mathcal{C}$ and morphisms $\underline{Hom_{\mathcal{C}}(X,Y)} = Hom_{\mathcal{C}}(X,Y)/P(X,Y)$ where $P(X,Y)$ is a class of all morphisms which factor through a projective.

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Hence, a morphism which factors through a projective is a trivial map and any composition of them is trivial as well. We refer to the $\underline{mod} - \hat{A}$ as the stable category of mod $- \hat{A}$. Modding these morphism give us a triangulated structure in $\underline{mod} - \hat{A}$ [Happel, 1988].

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What is the translation functor in $\underline{mod} - \hat{A}$? Remember, $\underline{mod} - \hat{A}$ has a enough injectives (projectives), for any module we have a short exact sequence

$$0 \to M \xrightarrow{f} I \to \Omega^{-1}(M) \to 0 \tag{19}$$

where $\Omega^{-1}(M)$ is the first cosyzgy of a module M (just cokernel of f). Its inverse is syzgy $\Omega(M)$ but it is not generally true that they are inverse for mod $-\hat{A}$.

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Ω^1 and Ω^{-1} forms an autoequivalence of categories $\Omega^{-1} : \mathcal{C} \to \mathcal{C}$ where the injectives/projectives (which are zero-objects in stable category) gets mapped to zero objects and this forms a distinguished triangle

$$X \to Y \to Z \to X[1] \tag{20}$$

Hence our stable category $\underline{mod} - \hat{A}$ using this trick (of modding out maps which factor through projective and injectives) gives an equivalence functor Ω^{-1} and we can easily construct a triangulation on $\mathcal{C}$ using objects, morphisms and the shift Ω^{-1}. This makes it to a triangulated category.

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Theorem (Happel,1988,91)

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Theorem (Happel,1988,91)

Stable category $\underline{mod} - \hat{A}$ *admits Auslander-Reiten triangles.*

Now we can to use this stable module category to study AR triangles.

Happel describes a functor in his 1988 textbook

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$\mu(X) \cong \mu(Y)$ which means $X \cong Y$

Happel's functor μ preserves the traingulated structure and hence, it maps triangles from $\underline{mod} - \hat{A}$ into $D^b(mod - A)$.

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There is also an important role of Serre functor.

Theorem (Reiten and Van den Bergh, 2001)

$D^b(mod - A)$ admits Auslander-Reiten triangles when $D^b(mod - A)$ admits Serre functor $S = - \otimes_A^L DA$.

For all the parts of the Happel's theorem and triangulated categories, one may see Happel, Dieter. Triangulated categories in the representation theory of finite dimensional algebras. Vol. 119. Cambridge University Press, 1988. Another reference is

- Happel, Dieter. "Auslander-Reiten triangles in derived categories of finite-dimensional algebras." Proceedings of the American Mathematical Society 112.3 (1991): 641-648.

Thank You.