# A (Quick) Note on Fourier Theory

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**Abstract**

We discuss the Fourier transformation of finite groups and Pontryagin duality. We discuss the motivation behind the character group.

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Our aim is to discuss the Fourier transform of an abelian group $G$ and how the vector space $V$ of representation of $G$ sheafifies over the character group $G^V$. The non-commutative version of this version for non-abelian is more difficult and lot more non-trivial due to difficulties in defining the dual group such that Fourier transform and inverse Fourier transform is vague. The standard reference for this note is [1].

Let us first see how quasi-coherent sheaves are developed for a certain commutative algebra $A$. For any commutative ring, we have the following $R-$module

$$R \to End(V) \tag{1}$$

where $V \in R-$mod. For this, one can define the spectral decomposition as following over a $\mathsf{Spec}\ R$

$$\overline{V} = V \otimes \mathcal{O}(U) \tag{2}$$

where $U \in \mathsf{Spec}\ R$ and $\overline{V}$ would be a sheaf here from this spectral decomposition [2, Section 01I6], see also [1,3] . There exists a canonical isomorphism between the R-mod $V$ and the sheaf $\overline{V}$ [2, Chapter 01H8]. $\mathsf{Spec}\ R$ is isomorphic to the locally ringed space $(\mathcal{O}_{\mathsf{Spec}\ R}, \mathsf{Spec}\ R)$, where $\mathcal{O}_{\mathsf{Spec}\ R}$ is the sheaf of rings and the pair is called affine scheme.

Anyways, for an abelian group $G$, we have the linear representation

$$\rho \colon G \to GL(V) \simeq End(V) \tag{3}$$

where $V$ is defined over some field $K$. (The map from G to $End(V)$ is a monoidal map.) To find what is the spectral decomposition in this example is ours job here. Answer is the character group.

Let us also revisit some character theory now[1]. A character $\mathcal{X}$ is the map

$$\mathcal{X} \colon G \to \mathbb{C}^\times \subset \mathbb{C} \tag{4}$$

where $\mathbb{C}^\times$ is the multiplicative ring, i.e. $\mathbb{C}/\{0\}$. Eq. (4) is basically the complex representation of the group $G$. The characters follow

$$\mathcal{X}(g)\mathcal{X}(h) = \mathcal{X}(gh), \quad g, h \in G \tag{5}$$

and they are also *orthogonal*. Character $\mathcal{X}$ send the element $g$ to the trace of the corresponding matrix in the representation

$$g \mapsto \mathcal{X}(g) = Tr(\rho(g)) \tag{6}$$

where $\mathcal{X}$ is a complex valued function. A character is called principal character if

$$\mathcal{X}(g) = 1 \tag{7}$$

otherwise it is non-principal character. The numbers of distinct characters can not exceed the order of any finite abelian group.

[1]For introduction to Character Theory of finite groups see the classical texts [4,5].

These characters are also invariant under the conjugacy class so

$$\mathcal{X}(hgh^{-1}) = \mathcal{X}. \tag{8}$$

Characters are also called **class functions**. To understand this, we need to understand that characters map to the the trace of the associated matrix in the representation $GL_n(F)$ for $G$. So basically $X(g) = Tr(\rho_V(g))$ and for the $\mathbb{C}$-endomorphism, we have $Tr(ab) = Tr(ba)$. Now

$$\begin{aligned} \mathcal{X}(hgh^{-1}) &= Tr(\rho_V(h)\rho_V(g)\rho_V(h^{-1})) && (9)\\ &= Tr(\rho_V(g)\rho_V(h)\rho_V(h^{-1})) && (10)\\ &= Tr(\rho_V(g)) = \mathcal{X}(g) && (11) \end{aligned}$$

This is the reason why it is also called conjugacy class or class functions.

Now let us consider an abelian group acting over a vector space $V$

$$\rho\colon G \to Aut(V) \equiv GL_n(F) \tag{12}$$

and the $\{\rho(g)\}_{g\in G}$ are the family of operators. Our job is to decompose $V$ in this spectral decomposition. For this linear representation over $G$, we have

$$g \mapsto \lambda(g) \tag{13}$$

such that $\lambda(g)\lambda(h) = \lambda(gh)$ where $g, h \in G$. This is what is called action of $G$ over the vector space $V$.

Let $v \in V$ be an eigenvector

$$g \cdot V = \mathcal{X}(g) \cdot v \tag{14}$$

where $\mathcal{X}$ is the character maps that we have discussed. The eigenvalues for these eigenvectors are given by the characters group

$$G^V = \mathsf{Hom}_{\mathsf{Grp}}(G, C^\times) \tag{15}$$

So, basically this is the spectrum for our spectral decomposition and we will do the spectral decomposition over $G^V$ now.

For the representation of group, we also have an inclusion which is a monoid map

$$G \to Aut(V) \subset End(V) \tag{16}$$

and

$$\mathsf{Hom}_{\mathsf{Monoid}}(G, Forget(End(V))) \tag{17}$$

We can now associate a $\mathbb{C}$ algebra with this group homomorphism

$$\mathsf{Hom}_{\mathsf{Monoid}}(G, Forget(End(V))) = \mathsf{Hom}_{\mathbb{C}}(\mathbb{C}G, End(V)) \tag{18}$$

where $G$ is finite as always and

$$\mathbb{C}G = \{\sum_{g \in G} F(g) \cdot g\} \tag{19}$$

where $f$ is a complex function from the group, i.e. $F\colon G \to \mathbb{C}$. The special thing about $F$ is that it associates a canonical element to every $g \in G$ such that $F$ is 1 at $G$ otherwise it is 0. So the canonical elements $g_F$ are

$$g_F = 1 \cdot g \in \mathbb{C}G \tag{20}$$

It is easy to see that the algebra structure of $\mathbb{C}G$ comes from the convolution of these elements $g_F$

$$g_F \star h_F = gh_F \tag{21}$$

and this algebra $(\mathbb{C}G, *)$ is commutative algebra iff $G$ is abelian. Basically, we have taken the action of $G$ and replaced it with the associated group $\mathbb{C}G$ and its action on $V$, i.e. $\mathbb{C}G \circlearrowright V$.

This is what we were hunting for and we will sheafify the representation of $G$ over the spectrum

$$\mathsf{Spec}\ (\mathbb{C}G, *) \tag{22}$$

and this is just the character group $G^V$

$$\mathsf{Spec}\ (\mathbb{C}G, *) = G^V. \tag{23}$$

This is the *Fourier transform* that we were looking for and we just sheafifes over this spectrum and apply these methods to understand the quantum theory.

What is important to understand that the Fourier transform changes from the $G$ to its characters group $G^V$ and we sheafifies over this $G^V$ and the associated algebra is just $(\mathbb{C}G, *)$. In case of seeing it simply, we have the following integral transform

$$f = \int_{\alpha \in G} \hat{f} \cdot \mathcal{X}_\alpha \tag{24}$$

where $\mathcal{X}_\alpha$ is just the character representation of $\alpha$. So basically, there is a very interesting duality between the group $G$ and the character group $G^V$.

We can call this $G^V$ the dual abelian group of $G$ and $G^V$ is abelian. Under this, we can canonically embed $G$ in $\hat{\hat{G}}$. This is basically

$$g \longrightarrow \{\mathcal{X} \mapsto \mathcal{X}(g)\}, \quad g \in G \tag{25}$$

which is an isomorphism.

*Update on 20 Jul: We will discuss the Pontryagin Duality and Lurie's Categorification of Fourier theory in a later version of this same note.*

## References

[1] J. Van Dyke and D. Ben-Zvi, “Between electric-magnetic duality and the Langlands program.”.

[2] T. Stacks project authors, “The Stacks project.” https://stacks.math.columbia.edu, 2024.

[3] J.-P. Serre, “Faisceaux algébriques cohérents,” *Annals of Mathematics* **61** no. 2, (1955) 197–278.

[4] I. M. Isaacs, *Character theory of finite groups*, vol. 69. Courier Corporation, 1994.

[5] J.-P. Serre *et al.*, *Linear representations of finite groups*, vol. 42. Springer, 1977.

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