QUIVER MODULI, MOTIVIC HALL ALGEBRA, AND AUSLANDER-REITEN TRIANGLES

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Abstract. In this note, we discuss how a quiver moduli stack reveals homological and representation-theoretic data of a quiver. The motivic Hall algebra provides the bridge.

"Now, that is not wisely asked; has it not been told thee, that the gods made a bridge from earth, to heaven, called Bifröst?" — Gylfaginning, Prose Edda

We will start with the definition of the Grothendieck ring of varieties $K_0(\mathrm{Var}_k)$. Let k be a field. We define a free abelian group generated by isomorphism classes $[X]$ where X is a variety, with some relations, known as the cut and paste relation (or scissors relation), for every closed subvariety Y of X, we write $[X] = [Y] + [X\backslash Y]$. The ring structure is given by $[X] \cdot [Y] \coloneqq [X \times_k Y]$.

We define the Lefschetz motive as $\mathbb{L} \coloneqq [\mathbb{A}_k^1]$ and define a localized ring $\mathcal{M} \coloneqq K_0(\mathrm{Var}_k)[\mathbb{L}^{-1}]$. The class $[X] \in K_0(\mathrm{Var}_k)$ is called *motive/class* of variety $X \in (\mathrm{Var}_k)$. For example, the class $[\mathbb{A}_k^n] = [\mathbb{A}_k^1 \times \mathbb{A}_k^2 \cdots \times \mathbb{A}_k^n] \cong \mathbb{L}^n$. Another example is to consider $\mathbb{P}^n$ which is

$$\mathbb{P}^n = \mathbb{A}^n \sqcup \mathbb{P}^{n-1} \tag{1}$$

$$= \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \cdots \sqcup \mathbb{A}^0 \tag{2}$$

so

$$[\mathbb{P}^n] = 1 + \mathbb{L} + \mathbb{L}^2 + \cdots + \mathbb{L}^n \tag{3}$$

which can be written as a geometric series (when $(\mathbb{L} - 1)$ is inverted)

$$[\mathbb{P}^n] = \frac{\mathbb{L}^{n+1} - 1}{\mathbb{L} - 1} \tag{4}$$

Due to the result of [1], we know that inverting $\mathbb{L}$ kills some non-zero classes, as $\mathbb{L}$ is a known zero-divisor in $K_0(\mathrm{Var}_k)$ for characteristic zero k. Hence, there is a loss of information in the localised ring. In other words, the map $K_0(\mathrm{Var}_k) \to K_0(\mathrm{Var}_k)[\mathbb{L}^{-1}]$

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is not injective for the field of characteristic zero. What follows in the next couple of paragraphs is for expository purposes.

We define a virtual motive for $[X]$ if X is irreducible

$$[X]_{\text{vir}} \coloneqq (-\mathbb{L}^{-\dim X/2})[X] \in \mathcal{M} \tag{5}$$

which is a dimension-dependent twist by $(-\mathbb{L}^{-\dim X/2})$. The square root of $\mathbb{L}$ and its multiplicative inverse are important in the theory of the Donaldson-Thomas invariant. For example, for $\mathbb{P}^1$, the virtual motive $[\mathbb{P}^1]_{\text{vir}} = -(\mathbb{L}^{-1/2} + \mathbb{L}^{1/2})$. But here, technically, it is just a normalisation because there is no singularity in $\mathbb{P}^1$. For, another non-trivial virtual class, $[\mathbb{P}^2]_{\text{vir}} = -(\mathbb{L}^{-1} + 1 + \mathbb{L}^1)$. The first non-trivial and non-smooth example would be to consider $[\mathbf{Hilb}^2(\mathbb{A}^3)]_{\text{vir}}$, see [2], but we would not do it explicitly here.

Now, if $k = \mathbb{F}_q$, then a measure taking value in $\mathbb{Z}$ is just the counting measure $K_0(Var_k) \to \mathbb{Z}$, $[X] \mapsto \#X(\mathbb{F}_q)$. The Euler characteristic is also a measure (assume $k = \mathbb{C}$) $\chi_{\mathbb{C}} : K_0(var_{\mathbb{C}}) \to \mathbb{Z}$, $[X] \mapsto \chi_{\mathbb{C}}(X)$. And the Hodge-Deligne polynomial is also a measure encoding mixed Hodge numbers $K_0(Var_k) \to \mathbb{Z}[u, v]$, and this was the central theme of the birational Calabi-Yau discussion of string theory.

A *Quiver* Q is a multi-directed graph with a set of vertices Q_0 and a set of arrows Q_1. For example, A_3 is

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

and K_2 is

$$1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2$$

We write a representation V of Q which consists of finite-dimensional vector spaces V_i for each vertex $i \in Q_0$ and a linear map $T_\alpha : V_i \to V_j$ for $\alpha : i \to j$ in Q_1. Two representations $((V_i), (T_\alpha))$ and $((V_i'), (T_\alpha'))$ are equivalent if there exists φ_i and φ_j such that the following commutes for each V_i and T_α

$$\begin{array}{ccc} V_i & \xrightarrow{T_\alpha} & V_j \\ {\scriptstyle \varphi_i}\downarrow & & \downarrow{\scriptstyle \varphi_j} \\ V_i' & \xrightarrow[T'\alpha]{} & V_j' \end{array}$$

We define the dimension vector $d \in \mathbb{N}Q_0$ of a quiver representation as a tuple $d = (\dim V_i)_{i \in Q_0}$. If we fix vector spaces V_i on Q, then we write the representation space

$$R_d(Q) \coloneqq \bigoplus_{\alpha : i \to j} Hom(V_i, V_j) \tag{6}$$

so that the vector spaces V_i are just points in an affine space. If $V_i \cong k^{d_i}$, then $R_d(Q) \cong \mathbb{A}_k^{\sum_{\alpha:i\to j} d_i d_j}$.

Classification of representations has been a central theme in the study of quivers. For Dynkin quivers (those with finite representation type), it is given by Gabriel's theorem [3]. But a more algebro-geometric approach for quiver representations is presented using the following. Fix a dimension vector d, and we define a reductive algebraic group

$$G_d = \Pi_{i\in Q_0} GL(V_i) \tag{7}$$

which acts on $R_d(Q)$ by a Q-graded analogue of conjugation

$$(g_i) \cdot (f_\alpha) \coloneqq (g_j f_\alpha g_i^{-1})_{\alpha:i\to j} \tag{8}$$

and we will identify this as a base-change action. The orbits of this action will be the isomorphism class of representations of a fixed d of the quiver Q.

The topology on the orbit space $R_d(Q)/G_d$ is not really helpful, as some non-isomorphic representations can lie in orbit-closure relations. Let us define a quotient (algebraic) stack $\mathcal{M}_d = \left[\frac{R_d(Q)}{G_d}\right]$, which will be a moduli stack. But there is an interesting line of thought about motivic invariants that begins with the consideration of a formal power series. For a symmetric matrix A defining a quiver, we write a partition function[1] or a formal power series, following [4]

$$P_A(x) \coloneqq \sum_{d\in\mathbb{N}Q_0} \frac{[R_d(Q_A)]_{\text{vir}}}{[G_d]_{\text{vir}}} x^d \tag{9}$$

where $x^d = x_1^{d_1} \cdot \ldots \cdot x_n^{d_n}$ are monomials. Note that the moduli stack $\mathcal{M}_d$ and the coefficients $[R_d(Q_A)]_{\text{vir}}$, $[G_d]_{\text{vir}}$ are different objects. While the former is geometric, the latter is more enumerative; precisely, the coefficients are virtual motivic classes that we have defined earlier. Both $\mathcal{M}_d$ and $P_A(x)$ can be used to count the quiver representations, and they lead to some very interesting stories. The formal power series is a motivic shadow of counting quiver representations. Let us stick to the moduli stack $\mathcal{M}_d$ in this paper, as our motivation is to pass to higher homological information.

Let us take a basic example of A_2 with linear orientation[2]

$$1 \xrightarrow{\alpha} 2$$

[1]This partition function formally takes value in a motivic quantum space.

[2]Reineke's [4] definition of $P_A(x)$ is for symmetric quivers, hence there is nothing for us to comment on that. Anyway, we will not discuss it in this note.

and say $d = (1, 1)$, then $R_d(Q) \cong \mathbb{A}^1_k$ and $G_d = GL_1(k) \times GL_1(k) \cong \mathbb{G}^m \times \mathbb{G}^m$ which acts by $(g_1, g_2) \cdot (\lambda) = g_2 g_1^{-1} \lambda$. There are exactly two orbits for this example, the one corresponding to the split representation $S_1 \oplus S_2$ and the one corresponding to the indecomposable (projective) representation P_1. This is, in fact, true that A_2 has three indecomposables which are, as represented by dimension vectors, $S_1 = (1, 0)$, $S_2 = (0, 1)$ and $P_1 = (1, 1)$. But the moduli stack $\mathcal{M}_d = \left[\frac{\mathbb{A}^1_k}{(\mathbb{G}_m \times \mathbb{G}_m)}\right]$ also remembers the stabilisers at points and records their automorphisms as different strata. Another reason for this moduli stack rather than a coarse quotient or orbit space will become clear in our next discussion, which is about the motivic Hall algebra.

We wish to understand how quiver representations connect with each other and what the structure of the category of representations $\mathrm{Rep}_k(Q)$ is. For this, we would like to study the extensions of representations, like $Ext^1_{\mathrm{Rep}_k(Q)}(A, B)$

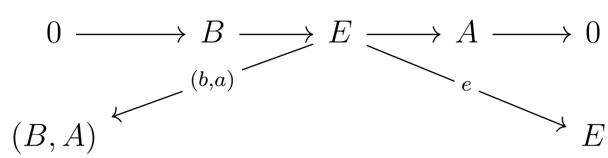
$$0 \to B \to E \to A \to 0. \tag{10}$$

We define $\mathcal{M} = \coprod_{d \in \mathbb{N}Q_0} \mathcal{M}_d$ of all the finite dimensional representations of Q. Formally, we define $\mathcal{M}^{(n)}$ as the moduli stack of n-flags of coherent sheaves on a base scheme [5, 6]. Let us take $\mathcal{M}^{(2)}$, which will be the stack of short exact sequences in $\mathrm{Rep}_k(Q)$ in the Joyce-Bridgeland sense in our quiver setting. Higher n-flags also exist, but we are not interested in them here; for instance, $\mathcal{M}^{(3)}$ will be a moduli stack of 2−steps filtrations.

Let us work with stacks which are locally of finite type. We define a correspondence of stacks

$$\begin{array}{ccc} \mathcal{M}^{(2)} & \xrightarrow{e} & \mathcal{M} \\ {\scriptstyle (b,a)}\downarrow & & \\ \mathcal{M} \times \mathcal{M} & & \end{array}$$

where the morphisms a, b, e takes a short exact sequence (10) to its constituent objects



The fiber of (b, a) over $(B, A) \in \mathcal{M} \times \mathcal{M}$ is the quotient moduli stack of extensions $[\mathrm{Ext}^1_{\mathrm{Rep}_k(Q)}(A, B)/\mathrm{Hom}_{\mathrm{Rep}_k(Q)}(A, B)]$ where $\mathrm{Hom}_{\mathrm{Rep}_k(Q)}(A, B)$ remembers the automorphism data of objects in extensions. The fiber[3] of e over E is the moduli of sub-representations of E, which parametrises the short exact sequences with a fixed

[3]In the sheaf-theoretic viewpoint, the fiber of b over $B \in \mathcal{M}$ is a Quot scheme.

E. Some comments about the nature of morphisms a, e, b can be found in [6]. Note that we have slowly progressed towards the homological side of $\mathrm{Rep}_k(Q)$ from the geometric knowledge about the same. More will be supplied by the motivic Hall algebra for our study.

Recall the moduli stack $\mathcal{M} = \bigsqcup_{d \in \mathbb{N}Q_0} \left[\frac{R_d(Q)}{G_d}\right]$. We define the relative Grothendieck group $K_0(St/\mathcal{M})$ to be the free abelian group generated by the morphisms of stacks $[X \xrightarrow{f} \mathcal{M}]$ where X is an algebraic stack, modulo cut-and-paste relations

$$[X \xrightarrow{f} \mathcal{M}] = [Y \xrightarrow{f_{|Y}} \mathcal{M}] + [U \xrightarrow{f_{|U}}] \tag{11}$$

for a closed substack $Y \subset X$ and $U = X \backslash Y$. We define *motivic Hall algebra* for $\mathrm{Rep}_k(Q)$ to be

$$\mathrm{Hall}_{\mathrm{mot}}(\mathrm{Rep}_k(Q)) := K_0(St/\mathcal{M}) \tag{12}$$

where the product is defined by the correspondence of stacks $\mathcal{M} \times \mathcal{M} \leftarrow \mathcal{M}^{(2)} \rightarrow \mathcal{M}$ that we discussed earlier. Explicitly, we have

$$[X_1 \xrightarrow{f_1} \mathcal{M}] \ * [X_2 \xrightarrow{f_2} \mathcal{M}] = [Z \xrightarrow{e \circ h} \mathcal{M}] \tag{13}$$

where h is given the Cartesian square

$$\begin{array}{ccccc} Z & \xrightarrow{\quad h \quad} & \mathcal{M}^{(2)} & \xrightarrow{\ e\ } & \mathcal{M} \\ \downarrow & & \downarrow{\scriptstyle (b,a)} & & \\ X_1 \times X_2 & \xrightarrow{(f_1 \times f_2)} & \mathcal{M} \times \mathcal{M} & & \end{array}$$

Hence, we claim that X_1 and X_2 are the families of subobjects and quotient representations in $\mathcal{M}$. We pullback $X_1 \times X_2$ to $\mathcal{M}^{(2)}$ along (b, a) and then pushforward along e to $\mathcal{M}$, and to a 'first' approximation as in [7], that the Hall product of two such families is given by taking their universal extension. So motivic Hall algebra $\mathrm{Hall}_{\mathrm{mot}}(\mathrm{Rep}_k(Q))$ packages all the short exact sequences in $\mathrm{Rep}_k(Q)$.

There are several Hall algebras formalisms: cohomological Hall algebras (CoHA, finitary Hall algebras, Hall algebras of constructible functions, but following Bridgeland [7], all of these Hall algebras can be thought of as different ways of taking "(co)homology" of the moduli stack of objects in an abelian category with convolution product provided by extension correspondence. In fact, our motivic Hall algebra is also a cohomology of a moduli stack in a sense, but we would not discuss it here. The motivation to do a Hall algebra varies through mathematics and physics; for instance, CoHA has been used in Reineke's program to understand Donaldson-Thomas invariants, stability conditions, and quantum torus algebra [4], and it has been used in physics to study BPS states as well [8–11].

There have been attempts to understand the derived Hall algebra [9, 12, 13]. But our motivation and purpose differ from those of derived Hall algebras. The passage from geometric information to our moduli stack leads to the structure of the abelian category $\mathrm{Rep}_k(Q)$ and thus the homological information therein. We know that there is an equivalence of categories $\mathrm{Rep}_k(Q) \simeq \text{Mod-}kQ$ where Mod-kQ is the category of (finite-dimensional) right kQ-modules. Similar to Auslander-Reiten (AR) sequences in Mod-kQ, we study the Auslander-Reiten triangles in $D^b(\text{Mod-}kQ)$. AR sequences in Mod-kQ are *almost split sequences* $Ext^1(M, \tau M)$

$$0 \to \tau M \to E \to M \to 0 \tag{14}$$

where τ is the AR translate, M and τM are indecomposables, and M is a non-projective module. For A_2, we have an AR sequence

$$0 \to S_1 \to P_1 \to S_2 \to 0 \tag{15}$$

In $D^b(\text{Mod-}kQ)$, an AR triangle is

$$\tau M \to E \to M \to \tau M[1] \tag{16}$$

where [1] denotes the shift (an automorphism of the category). But studying such triangles is non-trivial. For this, we find a triangulated category which is equivalent to $D^b(\text{Mod-}kQ)$ and study the AR triangles in that category. We know that Mod-kQ is a hereditary algebra and hence, finite global dimension (in fact, one), but in general, for any finite-dimensional algebra of finite global dimension, we have the following theorem due to Happel [14], also see [15].

Theorem 1 (Happel)**.** Let A be a finite dimensional $k-$algebra and A has finite global dimension (gl.dim $< \infty$). There exists a triangle equivalence between categories defined by a functor

$$\mu : D^b(\text{mod-}A) \to \underline{\text{mod-}}\hat{A} \tag{17}$$

where $\underline{\text{mod-}}\hat{A}$ is the stable module category over repetitive algebra $\hat{A}$.

We presented [16] that how AR triangles are easy to understand in a stable module category $\underline{\text{mod-}}\hat{A}$ which is a triangulated category[4] and under this functor,one can study AR triangles in $D^b(\text{mod-}A)$. Because it is easier to understand the shift functor (which happens to the cosyzgy functor Ω^{-1}) in $\underline{\text{mod-}}\hat{A}$.

The stratification data in $\mathcal{M}_d$ contains both split and non-split representations, namely $S_1 \oplus S_2$ and P_1, but the motivic Hall algebra distinguishes between the split and non-split in the data of extensions, which we need for AR sequences, but AR

[4]Happel shows [14, 16] that $\underline{\text{mod-}}\hat{A}$ is a Frobenius category and a stable category of a Frobenius category is a triangulated category.

theory further organises these non-split extensions to almost split extensions [17]. So, motivic Hall algebra does not produce AR sequences itself, but it makes the almost split sequences arising from our initial geometric data $\mathcal{M}$ more visible.

Our point in this note was that quiver moduli stacks contain more than just counting. At motivic Hall algebra, it already knows about the structure of extensions on $\mathrm{Rep}_k(Q)$. Once we recognise it, we can pass to AR theory and then to AR triangles in $D^b(\text{Mod-}kQ)$ which can be studied more concretely, due to Happel, in a stable module category over repetitive algebra of Mod-kQ. Thus, the passage from quiver moduli to motivic Hall algebra to AR theory can be thought of as a journey from enumerative geometry to homological representation theory. We chose AR theory as a specific interest; we can generalise this bridge to other homological interests. This line of thought has a lot to say about the algebra of BPS states.

We can ask a natural question whether a converse passage is true or interesting. Given a triangulated structure on the derived (or stable) category, how much of the quiver-theoretic model and its geometry can be recovered? But we do not pursue this thought here.

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