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Revering Musings on de Sitter and Holography

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Abstract

In these notes, we will discuss recent advancements in the area of de Sitter quantum gravity and de Sitter holography. We review the classical de Sitter and the asymptotic structure. Recent developments in the computation of entanglement entropy in de Sitter are also discussed. Quantum gravity in de Sitter has been an influencing subject; we review the recent work carried out in this area that builds the Hilbert space of de Sitter by considering the WDW solutions on late time slices. We also review the recent influence of von Neumann algebra in the area of de Sitter and black holes, where it was found that type II_1 is the von Neumann algebra for states of de Sitter static patch as observed by an observer on worldline.

Contents

1 Introduction

Two of the proudest achievements of theoretical physics in the last few decades have been the identification of holography and entanglement entropy descriptions. Observational cosmology predictions [1] provided a motivation towards making sense of our universe in the form of an approximately de Sitter spacetime,. The possibility of quantum gravity and holographic description of de Sitter has been heavily motivated after Maldacena introduced the AdS/CFT, which has had very nice results relating to things like subregion duality, entanglement entropy, and many algebraic aspects. It is now of interest to see if there is a similar set of results in the de Sitter framework, and this has been motivated in a series of papers [2,3]. However, since the de Sitter boundary is not a timelike one at spacelike infinity, dS/CFT is not as convenient as the AdS counterpart. For that matter, there are various approaches to de Sitter holography, such as traditional dS/CFT, static patch holography, half-de Sitter holography, and dS/dS.

It is suggested that cosmology had an expanding inflationary era where for every observer in the asymptotically de Sitter spacetime, there is a cosmological horizon that creates a constraint as to what the observer can observe, for instance, the observer can not send a gravitational wave to $\mathcal{I}^+$. This cosmological horizon, while being very similar to the black hole horizon[1], has some distinct features which will be discussed in these notes. One important difference is that the cosmological horizon is sensitive to the observer's position while the black hole horizon is global.

If we indeed live in an asymptotically de Sitter spacetime, then it must have a clock that can act as an observable, but it is hard to understand what it means [5]. While we do not explicitly report on these issues [6, 7], the importance of clocks would be important in our discussion of the algebra of observables of a static patch in de Sitter.

Quantum gravity in de Sitter has been an interesting issue since it is slightly different from our other experiences. In this review, we will review a recent paper by the authors of [8, 9] which constructs a Hilbert space of de Sitter Wheeler-de Witt state solutions. The issue of de Sitter quantum gravity is old though. Partly because of the reason that there does not exist a spatial infinity, there does not exist de Sitter generators. In our discussion, in sec. 4.1, the (logarithm of) volume of Cauchy slice would serve as a clock.

The other interest of this review is the algebra of observables of spacetime. Recently, we have observed a right interest in studying algebras of observables for different scenarios. The advanced machinery of functional analysis and operator algebra surprises sometimes by finding its use in some common information problems of physics, which is exactly the case for the algebra of observables for the de Sitter static patch. We would review the paper [10] that argues that the algebra of observables for an observer in the de Sitter static patch is given by type II_1.

The paper has been organized as follows. In sec. 2, we start with some preliminaries

[1]For example, both horizons have entropies associated to them, see [4].

about the geometry of de Sitter. In its subsection 2.2, we discuss the asymptotic symmetries of de Sitter and isometries. We also see how to do the conformal completion of spacetimes with positive cosmological constant. In sec. 3, we discuss the psuedo entropy (and timelike entanglement entropy) that has recently been looked at by many authors. In that very section, we also explore different interpretations of entanglement entropy and connect between AdS/CFT with dS. Moreover, the bit-threads approach [11] has been discussed as well. In sec. 4, we discuss two ideas of de Sitter, namely Hilbert space of quantum gravity in de Sitter at late time and algebra of observables for an observer in de Sitter static patch. We will comment in detail on the issues of a general solution of Wheeler-de Witt states, the norm of such states and anomaly equations involved because of some functionals being used. In sec. 4.1.2, we discuss the algebraic approach to understanding a Hilbert space of de Sitter static patch based on [10]. Some other aspects and ideas about holography in de Sitter have been discussed briefly.

2 de Sitter Spacetime

The present motivation to study the de Sitter space in terms of holography and quantum gravity finds its roots in the success of such problems in AdS spacetimes. The case of AdS is somewhat more straightforward, partly due to the presence of a conformal boundary where one can study the bulk-boundary correspondence. In the case of de Sitter, such a conformal timelike boundary does not exist at spatial infinity – rather, the current statement of dS/CFT is that of some notion of holography from some Euclidean field theories that are live on some conformal boundaries in the late time slices; this presents some issues in regard to the quantities that were famously given a holographic description in the case of AdS/CFT, in particular that of entanglement entropy, which were given a holographic treatment by the works of Ryu-Takayanagi and Hubeny-Rangamani-Takayanagi, which identify the area of some minimal (extremal) surfaces as the entanglement entropy of some boundary region. In dS/CFT, the density matrix corresponding to this becomes non-hermitian; due to this, the notion of "pseudo-entropy" has been proposed for dS/CFT, which takes the form of a timelike entanglement entropy that Wick rotations can find between Euclidean AdS to dS.

On a more fundamental level, de Sitter has posed some interesting problems. For instance, following the works of Gibbons and Hawking, one can identify the de Sitter horizon with the same horizon thermodynamics as that of a black hole, to which one can attribute the gravitational entropy or the Bekenstein-Hawking entropy,

$$S = \frac{A(H_\Lambda)}{4G_N} \, , \tag{2.1}$$

where $A(H_\Lambda)$ corresponds to the cosmological horizon area. One could now ask if the thermodynamic interpretation of dS is related to some coordinate basis for which a holographic theory can be constructed. This is seen in the case of "static patch"[2] dS physics, where one starts by observing that the full spacetime cannot be encoded into the view of one observer in dS, we will justify and discuss this statement later. In the sense of a Penrose diagram, one says that for observers on the "north" or "south" poles (called the *pode* and *antipodes*), there are causal patches that are bounded by the future and past horizons. In fig-1, the top and bottom edges represent the null infinities $\mathcal{I}^{\pm}$, the diagonals represent the future and past horizons, and each horizontal surface represents a $D-$sphere. The region $A \cap B$ represents the set of all points that are causally accessible to signals sent from the pode, while the region $C \cap D$ represents the set of all points that are causally accessible to signals sent from the antipode. The region B is referred to as the causal patch for an observer at the pode, while the region D is the causal patch for an observer at the antipode.

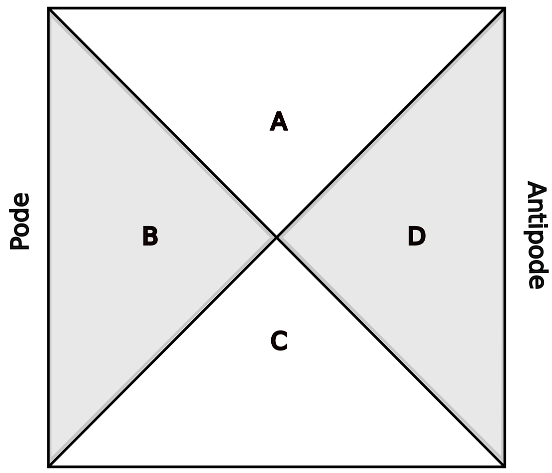

Figure 1: The causal structure of de Sitter. The regions $(A \cup B) \cap (B \cap C)$ is the causal patch associated to an observer at the pode, while the region $(A \cap D) \cup (C \cap D)$ is the causal patch associated to an observer at the antipode.

The "static patch" is the region B, which can be interpreted analogously to that of the Rindler patch. The metric for this is time-independent, with the interpretation that ∂_t is a Killing vector for the isometry of constant $t \to t + k$.

A cosmological horizon of de Sitter is very different from a black hole horizon. For an observer in de Sitter, a cosmological horizon is unique rather than a global object. The cosmological horizon of an observer moves with the observer. In the static patch coordinates, this horizon is at $r = 1$. Another interesting observation about cosmological spacetime that because different observers have effect on different horizons, the usual

[2]There exists a Killing vector field along the future timelike direction in the causally accessible region, which can be interpreted as a diffeomorphism generator in the bulk which shifts a geodesic to a future time. We will say that Hamiltonian H generates these time translations $t \to t + k$ and then because of this symmetry, the metric around the causally accessible region becomes time-independent. Hence the name "static patch".

theory of symmetry of black hole horizon do not work. One can do, however, write similar complementarity for de Sitter horizon [12] and among other things, analogously learn about the thermodynamics of the horizon [4].

In this section, we will motivate the geometric and thermodynamic aspects of classical de Sitter spacetime. We will start by reviewing the coordinates in which one can work in dS, particularly the global and Poincaré coordinates, and discuss the causal structure of dS. We will then discuss some aspects of Euclidean de Sitter spacetime, where we will look at the thermodynamic aspects of a Wick-rotated dS space. We would also look at the boundary of de Sitter and BMS group. The Lie algebra, which is $SO(D,1)$, for de Sitter is discussed as well.

2.1 de Sitter coordinates

One can start from a $D+1$ dimensional Minkowskian space and interpret de Sitter as hypersurface,

$$-X_0^2+\sum_{a=1}^{D} X_a^2 = l_{\text{dS}}^2 \ , \tag{2.2}$$

where one has $SO(D,1)$ isometry group. The constant l_{dS} is considered as the length scale for de Sitter[3]. We can get the global coordinates by the following redefinitions of the coordinates:

$$X^0 = \sinh t, \tag{2.3}$$

$$X^a = \cosh t \omega^a \ , \tag{2.4}$$

where (t,ω^a) are the global coordinates. By these definitions, and setting $l_{\text{dS}}=1$, the metric for de SItter in global coordinates becomes

$$ds^2 = -dt^2 + \cosh^2 t d\Omega_{D-1}^2 \ . \tag{2.5}$$

where $d\Omega_{D-1}^2$ corresponds to the angular components for the $D-1$ dimensional spherical line element. The static patch metric can be obtained via coordinate transformations as

$$ds^2 = -\left(1-r^2\right)dt^2 + \frac{dr^2}{(1-r^2)} + r^2 d\Omega_{D-2}^2 \ , \tag{2.6}$$

where one can imagine this as the coordinates defined around an observer sitting on the pode. One can also find other coordinate systems to deal with de Sitter physics; for instance, one could be interested in the planar coordinates, where the following coordinates are defined:

$$X^0 = \sqrt{1-r^2}\sinh t, \tag{2.7}$$

$$X^a = r\omega^a, \quad X^D = \sqrt{1-r^2}\cosh t \ , \tag{2.8}$$

[3]We will set $l=1$ throughout this review except for certain instances where we are explicitly dealing with Wick rotations involving the length scales, such as the case of Euclidean dS or AdS/dS/CFT sections.

which would give us the metric,

$$ds^2 = \frac{1}{\eta^2}\left(-d\eta^2 + \sum_{a=1}^{D-1} dx^a dx^a\right) , \tag{2.9}$$

where the η is the conformal time coordinate. Clearly, de Sitter has a conformal boundary at timelike infinity, and one can be tempted to try to find some holographic description exists similar to that of the AdS/CFT holography. In principle, using Wick rotations, one can see that essentially using double Wick rotations, one obtains the planar de Sitter patch. For the sake of showing this, we will retain the AdS and dS length scales l_{AdS} and l_{dS} – start by Wick rotating the Poincaré AdS patch,

$$ds^2 = \frac{l_{\text{AdS}}^2}{z^2}\left(-dt^2 + dz^2 + \sum_{a=1}^{D-2} dx^a dx^a\right) , \tag{2.10}$$

into a Euclidean AdS patch, and then using

$$z \longrightarrow i\eta \, , \tag{2.11}$$

$$l_{\text{AdS}} \longrightarrow -il_{\text{dS}} \, , \tag{2.12}$$

one recovers the planar de Sitter patch (2.9). In fact, this will be used in the next section, where we discuss the "pseudo" entanglement entropy in dS; timelike entanglement entropy in AdS can be recovered via these double Wick rotations For our purposes, it will be preferable to work in the global and Poincaré coordinates[4], which have the nice feature that they can be transformed into Euclidean AdS as seen before. This is particularly useful in the next section, where we will discuss timelike entanglement entropy and the holographic outlook of entanglement entropy.

2.2 Symmetries and Boundaries

We now defer our attention to symmetries of de Sitter spacetime and the boundary conditions. The latter is a more deeper subject, while the former answers the obvious geometrical questions of de Sitter. The boundary conditions are usually given at $\mathcal{I}^+$ (which resides at $\eta \to 0$), which is at the timelike infinity[5]. For symmetries in de Sitter, out first sight is the (bulk) isometry group for de Sitter metric Eq. (2.5), which is a semi-simple[6] Lie group $SO(D, 1)$. When viewed as a hyperboloid, the isometries

[4]The specific case of $D = 3$ is considered in other sections in this review, particularly in discussions involving de Sitter holographic entanglement entropy. For the sake of argument, we have specifically mentioned these coordinates in $D = 3$, but these can be generalized to D dimensions.

[5]To avoid any confusion, we mention that dS boundary is spacelike located at timelike infinity from the center of the spacetime. Similarly, for Anti-de Sitter, the boundary is located at spatial infinity while the boundary is timelike.

[6]The representations and characters of this group were studied extensively by Harish-Chandra [13,14], see also [15–17].

of Minkowski space that preserve the hyperboloid include rotations J_{ij} and boosts K_i. This gives $SO(D,1)$ with the exact number of generators, which is required to dub the de Sitter as a maximally symmetric solution, like AdS. Taking $r \to l_{dS}$ just tells us that the rotations give the Killing vectors of $D-1$ sphere. In contrast, the isometry group of AdS is $SO(D-1,2)$. Furthermore, the Lie group $SO(D,1)$ is isomorphic to the $(D-1)$-dimensional Euclidean conformal group which includes the dilation and conformal spatial transformations. For $D=4$, there is one dilation generator and four conformal spatial generators. One can also show that for a static patch worldline observer, there is a hidden $SL(2,\mathbb{R})$ as well [18], which is a subject for a deeper argument, see [19–23].

Let us comment briefly on the embedding features of de Sitter and Anti-de Sitter in a higher dimensional Minkowski's spacetime. The presence of a global Killing vector field is evident in both Minkowski's spacetime and Anti-de Sitter spacetime, which results in positive energy theorems. There does not exist a global Killing vector field for $\Lambda > 0$; see [24] for details. For this reason and partly because of the absence of a unique maximal hypersurface, one can not use the techniques of gauge-fixing to construct a Hilbert space of ref. [25]. There is a shared feature by de Sitter and Minkowski's spacetime, where holography can be explained by the irreducible unitary representations and principal series representations.[7] One more shared resemblance is found when we study the asymptotic symmetric group at $\mathcal{I}^+$, which is just the diffeomorphism group of $\mathbb{R}^3$ [26].

The other important symmetry to understand is the asymptotic symmetry. But it is preferential first to discuss what the boundary conditions look like for a conformal boundary in static patch de Sitter. If we compare de Sitter with other maximally symmetric spacetimes and their *asymptotics*, then for AdS, there is a well-defined spatial infinity where we define the boundary. This is well understood. For Minkowski space, the null infinity is well-defined as well. But the asymptotes for de Sitter are at timelike future and past infinity $\mathcal{I}^+$ and $\mathcal{I}^-$. We can start by asking what are the boundary conditions for asymptotically Anti-de Sitter space (see ref. [25, 27]) and guess the possible boundary conditions for de Sitter which solves many stability queries. Moreover, many attempts have formalized the boundary conditions in de Sitter [26, 28].

We will define the conformal completion for $\Lambda > 0$ spacetimes now. Let $(\tilde{M}, \tilde{g})$ be a solution of Einstein's equation with $\Lambda > 0$. We can not study the asymptotic points in this spacetime because there does not exist a point at infinity. But we can impose a conformal scaling of the metric $\tilde{g}$

$$\tilde{g} = \Omega^2 g \tag{2.13}$$

where g is a unique, up to a conformal factor, Riemannian metric on M. Ω is a smooth function, satisfies $\Omega \geq 0$, $\Omega = 0$ at $\mathcal{I}$ and $\Omega^{-2} T_{\mu\nu}$ falls off smoothly on $\mathcal{I}$. Solution $\tilde{g}$ must also obey Einstein's field equation for $\Lambda > 0$. A quick exercise is to check for the further scaling of Ω, $\Omega \to \omega^{-1}\Omega'$, where ω is smooth too and not zero even at $\mathcal{I}$. This kind of rescaling results in g' and it must satisfy the same Einstein's field equation. Thus we can introduce the conformal class of metrics for $(\tilde{M}, \tilde{g})$ for such completion as $[\gamma]$.

[7] SO(D-1,2) lacks the principal series representation, see Sec. 6 of [16].

Let (M, g) be a solution of de Sitter spacetime $dS^{\pm}$, where $\pm$ on dS would become apparent in a second, then the conformal infinity for (M, g) is given by the components of a Riemannian manifold $(\partial M, \gamma)$ which are $\mathcal{I}^+$ and $\mathcal{I}^-$. Usually, we have $M = \mathbb{R} \times \Sigma$ and Σ is diffeomorphic to $\mathcal{I}^{\pm}$. These conformal infinities behave spacelike, because of $\partial\Sigma$. One could show that for $d = 4, n = 3$, the fundamental group gives $|\pi_1(\mathcal{I}^+)| = \infty$ and $\mathcal{I}^- = \emptyset$, the only possibility of $\mathcal{I}^+$ is to be spherical 3-forms S^3. For this, $M = \mathbb{R} \times S^3$. For $(M, g^+) \in dS^+$, we put the boundary date $(g^{(0)}, g^{(n)})^+$ on $\mathcal{I}^+$ and let it determine, up to isometry, the global solution (M, g). One can either work on $\mathcal{I}^+$ for (M, g^+) by fixing $\mathcal{I}^- = \emptyset$ for (M, g^-) or on $\mathcal{I}^-$ for (M, g^-) by fixing $\mathcal{I}^+ = \emptyset$ for (M, g^+). The boundary data on the infinity is given from the Fefferman-Graham expansion (or Starobinsky asymptotic expansion) [29, 30]. The conformal completion depends of the sign of Λ.[8]

For a flat spacetime, the boundary is defined at null infinity with retarted coordinates. The asymptotic symmetries of this conformal metric generate an infinite-dimensional group called the BMS group, first studied by Bondi, Miesner and Sachs [31–33]. The (future) null infinity is generally more complex than the spatial infinity because the radiation can escape to the null infinity. We also wish to study such asymptotic symmetries for the boundaries $\mathcal{I}^{\pm}$ of de Sitter.[9]

To understand the dynamics of $\mathcal{I}^+$ (we only want to choose the future boundary $\mathcal{I}^+$ by fixing $\mathcal{I}^- = \emptyset$) we must first specify the boundary conditions. Ideally, we want to impose boundary conditions that allow gravitational wave evolution to $\mathcal{I}^+$. "Fefferman-Graham" gauge conditions give such solutions for late time [29]

$$\frac{ds^2}{l^2} = -\frac{d\eta^2}{\eta^2} + \frac{dx^i dx^j}{\eta^2}\left(g_{ij}^{(0)} + \eta^2 g_{ij}^{(1)} + \eta^3 g_{ij}^{(3)} + \cdots\right) \tag{2.14}$$

where the Einstein equation (for positive Λ) imposes constraints on the expansion. In Eq. (2.14), η is a solution of the conformal class $[\Omega]$ and $\eta \to 0$ gives solution for $\mathcal{I}^+$. These expansions are done around $g^{(0)} = \gamma$ and depend on the choice of the boundary metric $\gamma \in [\gamma]$. In particular, the constraint

$$tr(g^{(n)}) = 0 \tag{2.15}$$

is determined by γ, but otherwise it depends on M. The coefficients of power in expansion depend on $g^{(0)}$ and $g^{(n)}$ and this is the essential boundary data that we earlier

[8]For $\Lambda < 0$, any solution of form (M, g) has a conformal infinity at $\partial M = \mathbb{R} \times \partial\Sigma$ where Σ is a spacelike slice in M. For $\Lambda = 0$, any solution of form (M, g) has a conformal infinity at $\partial M = \mathbb{R} \times \partial\Sigma$, where Σ is a compact null hypersurface in M.

[9]One may ask why it is beneficial to study the asymptotic in a theory of gravity and how it differs from a local quantum field theory. An algebraic answer is the following. In a theory of gravity, the information is locally available on the boundaries without the loss of causality. We can take the boundary limit of operators in the bulk and define the algebra at the null infinity (for the flat space) and this is a better-known algebra than the algebra of the operators in the bulk. This can be done without assuming holography. While the algebras of local quantum field theories are usually ill-defined in the context of entropy.

defined on $\mathcal{I}^+$. In other words[10], ∂M of M is determined by $g^{(0)}$ and $g^{(n)}$. Furthermore, $g^{(n)}$ determines the $\langle T_{\mu\nu}\rangle$ of CFT living on $\mathcal{I}$.

We look for the diffeomorphisms of dS^+ metrics such that Fefferman-Graham form is preserved, and we will call allowed diffeomorphism as those under which $(g^{(0)}, g^{(n)})^+$ is invariant. The asymptotic symmetric group is just defined as a quotient

$$\mathrm{ASG} = \frac{\mathrm{Diff}}{\mathrm{Allowed\ Diff}}. \tag{2.16}$$

The non-trivial part of ASG is just the diffeomorphism group of $\mathbb{R}^3$ [26]. This group is essential to understand the charge conservation and the dynamics of $g^{(n)}$. For comparison, the asymptotic symmetric group for flat spacetime is the original BMS group.

3 Entanglement Entropy in de Sitter

AdS/CFT has been very successful in defining entanglement entropy. Here, we will review entanglement entropy.

3.1 Pseudo Entropy

We will start by first discussing what pseudo entropy is, since we will interpret timelike entanglement entropy throughout this section. Let $|\psi\rangle$ and $\langle\phi|$ be two pure states and the inner product be nonzero. We define the transition matrix,

$$\mathcal{T} \equiv \mathcal{T}^{\psi|\phi} = \frac{|\psi\rangle\langle\phi|}{\langle\phi|\psi\rangle}\ , \tag{3.1}$$

which satisfies that the trace is normalized to one. One can also see that the trace of the N^{th} power of the transition matrix also follows this trace property since one can find that $(\mathcal{T})^N = \mathcal{T}$ and therefore the traces must also be equal to 1. If we define a bipartitioned Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, we can define the reduced transition matrices w.r.t each subsystem. If we choose A, then we trace out $\mathcal{T}$ w.r.t B to define

$$\mathcal{T}_A = \mathrm{Tr}_B\left[\frac{|\psi\rangle\langle\phi|}{\langle\phi|\psi\rangle}\right]\ , \tag{3.2}$$

and for this one can define the N-th Rényi entropy in the usual way:

$$\mathcal{S}^N(\mathcal{T}_A) = \frac{1}{1-N}\log \mathrm{Tr}\left(\mathcal{T}_A^N\right)\ . \tag{3.3}$$

[10] It is also worth noting that $g^{(0)}$ corresponds to Drichlet boundary conditions and $g^{(n)}$ corresponds to Neumann boundary conditions. But this is mostly applicable to Euclidean AdS and sometimes we can also define a map $g^{(0)} \to g^{(n)}$ but that requires very specific settings.

Since the matrix $\mathcal{T}$ is not Hermitian in particular, the entropy defined by this matrix would take complex values. Due to the non-hermitian specific nature of this matrix (and $\mathcal{T}_A$), the complex values of (3.3) interpret this Rényi entropy as a pseudo-entropic quantity. Setting $N \geq 2$ and positive natural number values, one can establish that the N-th Rényi entropy can be expressed in terms of the eigenvalues of $\mathcal{T}_A$ as

$$\mathcal{S}^N(\mathcal{T}_A) = \frac{1}{1-N} \log \left(\sum_a \lambda_a \left(\mathcal{T}_A\right)^N \right) . \tag{3.4}$$

Since we do not take $\mathcal{T}$ to be hermitian, diagonalization is a non-trivial issue. As stated in [34], one can go around this by Jordan decomposition of $\mathcal{T}_A$ to arrive at (3.4), which we will not reproduce here. If one takes the $N \to 1$ limit of the N-th Rényi entropy, one gets the von Neumann entropy, which is interpreted as the pseudo-entanglement entropy defined w.r.t $\mathcal{T}_A$:

$$S_{vN}\left(\mathcal{T}_A\right) = \lim_{N \to 1} \mathcal{S}^N(\mathcal{T}_A) \Longrightarrow - \sum_a \lambda_a \left(\mathcal{T}_A\right) \log \left(\lambda_a \left(\mathcal{T}_A\right)\right) . \tag{3.5}$$

In the sense of quantum field theories, one can compute the pseudo entanglement entropy using the replica trick, using which one can find the N-th Rényi entropy in the path integral formulation. This is done by taking the path integral over manifolds for some Euclidean action $S[\varphi]$ (where φ is the field configuration), and consider the products $|\psi\rangle\langle\phi|$ and $\langle\phi|\psi\rangle$ as the path integrals giving the transition matrix $\mathcal{T}$. By bipartitioning the Hilbert space, one can get the reduced transition matrices $\mathcal{T}_A$ and the N-th power of the trace of this w.r.t A via the replica trick with N-copies. By considering the manifold describing the N-th power as Σ_N and the path integral over the manifold M as $Z(M)$, the N-th Rényi entropy in terms of $\mathcal{T}_A$ as

$$\mathcal{S}^N(\mathcal{T}_A) = \frac{1}{1-N} \log \left(\frac{Z(\Sigma_N)}{Z^N(\Sigma)} \right) , \tag{3.6}$$

where the manifold for $\langle\phi|\psi\rangle$ is denoted as Σ – the pseudo entanglement entropy becomes the $N \to 1$ limit of this.

Pseudo entropy has some basic features, essentially amounting to the following conditions choosing positive real values of N: if a state has no entanglement, the corresponding N-th Rényi entropy is equal to zero. Next, the N-th Rényi entropy computed w.r.t the reduced transition matrix $\mathcal{T}_A$ and the N-th Rényi entropy computed w.r.t. the reduced transition matrix $\mathcal{T}_B$ are equal to each other. Finally, choosing positive real values of N except unity, the N-th Rényi entropy satisfies $\mathcal{S}^N(\mathcal{T}_A) = \mathcal{S}^N(\mathcal{T}_A)^*$[11].

At this point we must note some important points; firstly, in general, one has to worry about which branch is picked in the consideration of the log-function. However, we have dropped a discussion on this as for now, but we will comment on this when

[11] Or equivalently, as noted in [34], one can impose a more general condition on the eigenvalues of $\mathcal{T}_A$ to state this equivalence.

discussing the equivalence between timelike entanglement entropy and pseudo entropy. Secondly, it must be noted that we have also not discussed in detail the relation between Rényi entropy in the transition matrices approach and the following relation:

$$S(\mathcal{T}_A) = -\,\mathrm{Tr}\left(\mathcal{T}_A \log \mathcal{T}_A\right) \ . \tag{3.7}$$

However, we will consider this to be the general definition of pseudo entanglement entropy, as we shall see in the next subsections. We will now discuss the cases of AdS/CFT and dS/CFT, and introduce timelike entanglement entropy in both the cases.

3.1.1 Ryu-Takayanagi

Naturally, in the continuum limit $\epsilon \to 0$, it is easy to see that the entanglement entropy becomes infinite due to UV divergences. The divergences can be captured in this UV limit as:

$$S_R = \gamma \frac{\text{Area of } \partial R}{\epsilon^{D-2}} + \text{less divergent terms} \ . \tag{3.8}$$

From [35–37], one has the following formula for entanglement entropy,

$$S_R = \frac{c}{6} \mathcal{A} \log \frac{\xi}{\epsilon} \ , \tag{3.9}$$

where c is the central charge of AdS, $\mathcal{A}$ is the number of boundary points and ξ is the correlation length. In our case, $\mathcal{A} = 2$, and $c = \frac{3R}{2G_N}$. Further, the role of a UV cutoff can be made clearer by noting that, in global coordinates, the boundary induces a divergence in the metric. The metric in these coordinates takes the form,

$$ds^2 = R^2 \left(-\cosh \rho^2 dt^2 + d\rho^2 + \sinh \rho^2 d\theta^2\right) \ . \tag{3.10}$$

Therefore, one introduces a cutoff in the conformal coordinate ρ_0, which corresponds to a UV cutoff in the dual CFT – one then sees that $\exp L \sim L/a$, where a is the UV cutoff and L is the total length with periodic identification. Following this, the Ryu-Takayanagi formula for calculating the entanglement entropy is of the form,

$$S_R = \frac{\text{Area of } \mathcal{X}_{RT}}{4G_N} \ , \tag{3.11}$$

where R is some boundary CFT subregion, and $\mathcal{X}_{RT}$ is a minimal surface whose area gives the entanglement entropy, called the Ryu-Takayanagi surface. One can explicitly check that this reproduces the correct entanglement entropy by computing the length of the geodesic, which would precisely coincide with (3.9).

There is also a covariant holographic entanglement entropy proposal, which we will not review here. However, the core aspect of these proposals is that there exists a *spacelike* geodesic connecting two distinct points on the boundary – in the case of de Sitter, such is not possible, and this indicates a complexity in finding a formula such as (3.11) for a dS/CFT. As we shall see in the next, we consider timelike geodesics instead of spacelike geodesic lengths to compute entanglement entropy with a holographic duality.

3.1.2 Timelike Entanglement Entropy in AdS/CFT

The basis of timelike entanglement entropy is that one can consider a timelike geodesic to find the entanglement entropy of some subsystem. Let R be a subsystem with some spacelike and timelike components X, T. Then, the entanglement entropy becomes

$$S[R(X,T)] = \frac{c}{3}\log\frac{\sqrt{X^2 - T^2}}{\epsilon} , \tag{3.12}$$

which gives the Ryu-Takayanagi result (3.11) when $T = 0$. On the other hand, one can set $X = 0$ to obtain *timelike entanglement entropy*, which is of the form [38–40]

$$S[R(T)] = \frac{c}{3}\log\frac{T}{\epsilon} + \frac{ic\pi}{6} . \tag{3.13}$$

One can define timelike entanglement entropy by a similar procedure as that of spacelike entropy. The starting point of timelike entropy can be found by considering a scalar field theory – in two dimensions, this motivates timelike entanglement entropy by the use of a Wick rotation. The partition function is of the form,

$$\mathcal{Z}[\phi] = \int \mathcal{D}\phi e^{iS[\phi]}, \tag{3.14}$$

where the field ϕ has the usual Lagrangian of a massless scalar field. We now want to Wick rotate the coordinates so that the t coordinate plays the role of space, and x plays the role of Wick rotated time. By further imposing some "real" time coordinate T condition, $x = iT$, the action for the scalar field takes the form ,

$$S = \frac{i}{2}\int dT dt \left[(\partial_T\phi)^2 + (\partial_t\phi)^2\right] , \tag{3.15}$$

and from this one gets the Hamiltonian can be found. The Hamiltonian can be written as

$$H = \frac{1}{2}\int dt \left[\pi^2 + (\partial_t\phi)^2\right] , \tag{3.16}$$

where $\pi = i\partial_x\phi$ is the conjugate canonical momentum to ϕ. Then, the partition function becomes $\mathcal{Z}[\phi] = \mathrm{Tr}\, e^{i\beta H}$ (where β is the periodicity), and the corresponding reduced density matrix can be seen to be non-hermitian. The entanglement entropy corresponding to this would be the timelike entanglement entropy (3.13). One can make a more sophisticated computation using the replica trick with Renyi entropy, which would recover the entropy (3.13), which we will review briefly below.

One can arrive at the timelike entanglement entropy discussed above by considering the Rényi entropy considering a subsystem R with endpoints $A(T_a, X_a)$ and $B(T_b, X_b)$, and taking the limit of $N \to 1$. With this, we arrive back at the previous expression (3.12),

$$S_R^{N\to 1} = \frac{c}{3}\log\frac{\sqrt{(X_b - X_a)^2 + (T_b - T_a)^2}}{\epsilon} , \tag{3.17}$$

and in the case when the subsystem does not have any spacelike separation components, one gets back to the previous timelike entanglement entropy expression,

$$S_R^{N\to 1} = \frac{c}{3}\log\frac{T}{\epsilon} + \frac{ic\pi}{6} \ .$$

Since this has a non-hermitian density matrix, timelike entanglement entropy can be given the interpretation of pseudo entropy, which will be more appropriately discussed in the case of dS/CFT below.

3.1.3 From AdS to dS/CFT timelike Entropy

In the case of dS/CFT, one would necessarily have to deal with a non-unitary dual CFT, and due to this one gets a complex-valued entanglement entropy. As seen above, one sees that timelike entanglement entropy also gives a complex contribution, and can be regarded as a pseudo entropy. Let us consider this in more detail in below.

The dS case can be highlighted similarly to that of the AdS case by looking at the Gubser-Klebanov-Polyakov-Witten (GKPW) prescription,

$$\Psi_{\mathrm{dS}}[\varphi] = \mathcal{Z}[\varphi] \ , \tag{3.18}$$

where φ acts as the generating functional on the boundary and gives the boundary condition for the fields on the future boundary:

$$\Psi_{\mathrm{dS}}[\varphi_0] = \int_{\varphi_0 \equiv \infty} \mathcal{D}\varphi \exp\left(iS_{\mathrm{dS}}[\varphi]\right) \Psi_0 \ , \tag{3.19}$$

where Ψ_0 is an initial state at $t = 0$ and we have the boundary condition φ_0 at the future asymptotic boundary. Then, one can see that the partition function $\mathcal{Z}[g, \varphi]$ takes complex values. In order to give this a geometric interpretation, start by considering the $D + 1$ dS in global coordinates,

$$ds^2 = l_{\mathrm{dS}}^2 \left(-dt^2 + \cosh^2 t d\Omega_D^2\right) \ . \tag{3.20}$$

We will now perform a Wick rotation by $t \to i\tau$ for considering the Hartle-Hawking state, which gives us the Euclidean de Sitter space,

$$ds^2 = l_{\mathrm{dS}}^2 \left(d\tau^2 + \cos^2 \tau d\Omega_D^2\right) \ . \tag{3.21}$$

In general, the central charge of the D-CFT has a complex value in the dS/CFT framework. To see this better, one can consider the relation between the AdS and dS length scales, given by

$$l_{\mathrm{AdS}} \longrightarrow -il_{\mathrm{dS}} \ , \tag{3.22}$$

from which one gets the central charge of the CFT in dS; from the Brown-Henneaux central charge formula, we know that the central charge is $c_{\text{AdS}} = \frac{3l_{\text{AdS}}}{2G_N}$, which gives the central charge for dS_3/CFT_2,

$$c_{\text{dS}} = \frac{3R_{\text{dS}}}{2G_N} \, , \tag{3.23}$$

and we have $c = -ic_{\text{dS}}$. Overall, it is clear that in the dS/CFT framework, the dual CFT is non-unitary. Indeed, one can arrive at the timelike entanglement entropy in dS by a double Wick rotation, defined by $R_{\text{AdS}} = -iR_{\text{dS}}$, $z = -i\eta$, $t = -ix$, where we have transformed from the Poincaré de Sitter coordinates,

$$ds^2 = R_{\text{dS}^2}\left(\frac{-d\eta^2 + d\tau^2 + dx^2}{\eta^2}\right) \, . \tag{3.24}$$

With this in mind, one gets the timelike entanglement entropy in de Sitter space as:

$$S[R] = -i\frac{c_{\text{dS}}}{3}\log\frac{x}{\epsilon} + \frac{\pi c_{\text{dS}}}{6} \, . \tag{3.25}$$

One can easily see that this entropy has the exact interpretation as that of pseudo entropy. We could now be interested in the higher-dimensional notions of timelike entanglement entropy, which we will discuss below.

In the Poincaré dS_{D+1} coordinates, which would be given by the metric

$$ds^2 = R^2_{\text{dS}}\left(\frac{-d\eta^2 + dt_E^2 + d\mathbf{x}^2 + dy^2}{\eta^2}\right) \, , \tag{3.26}$$

where t_E is related to t as $t \longrightarrow -t_E$. As considered in [39], we will consider the subsystem R on a $y = 0$ slice with radius $T/2$ and introduce a radial coordinate $\mathbf{r} = \sqrt{t_E^2 + \mathbf{x}^2}$. We then consider a timelike surface given by η, $\mathbf{r}$ and T values, and the area of this surface (including the real part) by $-\eta^2 + \mathbf{r}^2 = T^2/4$ (and $\eta^2 - \mathbf{r}^2 = T^2/4$ respectively) generates the pseudo entropy, which comes in the case-specific values of even or odd D:

$$S[R] = \frac{R^{D-1}}{4G_N^{D+1}}\left[\left(\text{vol}(\mathbb{S}^{D-2})\right)\frac{\sqrt{\pi}\Gamma\left(\frac{D-1}{2}\right)}{2\Gamma\left(\frac{D}{2}\right)} + i\mathcal{D}\right] \, , \tag{3.27}$$

where $\mathcal{D}$ is given by

$$\begin{cases} \sum_{a=0}^{\frac{D-3}{2}} \binom{\frac{D-3}{2}}{a} \frac{1}{D-2a-2}\left(\frac{T}{2\epsilon}\right)^{D-2a-2} & \text{odd } D \, , \\ \sum_{a=0}^{\frac{D-3}{2}} \binom{\frac{D-3}{2}}{a} \frac{1}{D-2a-2}\left(\frac{T}{2\epsilon}\right)^{D-2a-2} + \frac{\Gamma\left(\frac{D-1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{D}{2}\right)}\log\frac{T}{2\epsilon} & \text{even } D \, . \end{cases} \tag{3.28}$$

In the sense of the Wick rotations considered previously to arrive at timelike entanglement entropy, one can start by the length scale transformations and the T transformations, and define the geometric rotation

$$\text{vol}\left(\mathbb{S}^{D-2}\right) \longrightarrow (-i)^{D-2}\text{vol}\,\left(\mathbb{H}^{D-2}\right) \, , \tag{3.29}$$

from which one gets the higher dimensional case in EAdS/CFT as discussed previously.

The real part of this is given by [38]

$$\frac{R_{\mathrm{dS}}^{D-1}\pi^{D/2}}{4G_N^{D+1}\Gamma\left(\frac{D}{2}\right)}\ , \tag{3.30}$$

which in the $D=2$ case becomes $\frac{\pi R_{\mathrm{dS}}}{4G_N^3}$, and since the central charge is given by (3.23), we get the real part $\frac{\pi c_{\mathrm{dS}}}{6}$. This real part also has a nice feature that this is related to the de Sitter entropy formula,

$$S=\frac{(2R_{\mathrm{dS}})^2\pi^{\frac{D-1}{2}}}{\Gamma\left(\frac{D-1}{2}\right)G_N} \tag{3.31}$$

by a factor of $\frac{1}{2}$.

3.2 de Sitter Ryu-Takayanagi and Bit Threads

The general issue with trying to formulate a de Sitter Ryu-Takayanagi formula is that one has a boundary at $\mathcal{I}^+$, due to which one gets a pseudo entropy as detailed above. However, the situation can be somewhat simplified by considering static patch de Sitter, where the boundaries for the pode-antipode systems are the corresponding stretched horizons respectively. In [41], a de Sitter formulation of the Ryu-Takayanagi prescription was introduced in static patch.

The idea is that one can start by assuming that the entanglement entropy of the pode-antipode system can be described by the area of some minimal surface that is homologous to the corresponding boundaries – the stretched horizon corresponding to the respective component. For the sake of discussion, we will consider the pode side, although the same can be stated for the antipode as well. One can formulate something as follows: take the pode-antipode system, and find a minimal surface $\mathcal{X}_{\mathrm{dSRT}}$ that is homologous to the boundary corresponding to either component – the, the entanglement entropy is

$$\frac{\text{Area of }\mathcal{X}_{\mathrm{dSRT}}}{4G_N}\ . \tag{3.32}$$

However, this has a very clear issue: the homologous condition is too weak, since one can simply take the surface and "flow" so as to get to zero area, implying that the entanglement entropy between the pode-antipode system is zero. One can go around this issue by constraining $\mathcal{X}_{\mathrm{dSRT}}$ to lie between the stretched horizons. Then, this indicates that the area of the surface $\mathcal{X}_{\mathrm{dSRT}}$ is the horizon area, giving the Gibbons-Hawking entropy,

$$S=\frac{\text{Area of dS horizon}}{4G_N}\ . \tag{3.33}$$

One can now give this a maximin formulation by maximizing over Cauchy surfaces cutting the stretched horizons after finding minimal area surfaces (which lie on the stretched horizons). However, this is still not enough as a prescription for calculating entanglement entropy. Following [11], we will discuss the use of "bit threads" as introduced by [42] to describing entanglement entropy in static patch de Sitter.

Bit threads were introduced by Freedman and Headrick [42] as another formulation of Ryu-Takayanagi prescription, where one considers *bit threads* of some fixed width $1/4G_N$ as sourced from the boundary. In the AdS/CFT sense, this can be described as follows: take a boundary subregion ∂R, and let it source bit threads[12]. Then, the Ryu-Takayanagi formula can be formulated in terms of bit threads by considering max # of bit threads leaving ∂R and minimizing the area of a surface homologous to ∂R anchored to the boundary:

$$S(\partial R) = \frac{1}{4G_N} \max_{\#\text{threads}} \leftrightarrow \min_{\mathcal{X}\sim\partial R} \text{Area of } \mathcal{X} \,, \tag{3.34}$$

From max-flow-min-cut, the minimizing term is the same as maximizing the flows v originating from ∂R, becoming redundant and reducing to

$$\max_{\#\text{threads}} \int_{\partial R} v \,. \tag{3.35}$$

Following the covariant holographic entanglement entropy formula by Hubeny, Rangamani and Takayanagi [43], one can attempt to reproduce the HRT prescription in terms of bit threads. In fact, this can also be done by considering the maximin prescription due to Wall [44]. The HRT in maximin terms is of the form

$$S(\partial R) = \max_{\partial R\subset\Sigma} \min_{\mathcal{X}\sim\partial R} \text{Area of } \mathcal{X} \,. \tag{3.36}$$

Then, one can apply a Lorentzian version of the max-flow-min-cut principle by maximizing the flows v on Σ:

$$\max_{\Sigma\sim\partial\partial R} \text{vol}(\Sigma) = \min_{\text{flows } v} \int_{\partial R} n\cdot v \,. \tag{3.37}$$

.where by $\partial\partial R$ we mean the boundary of the boundary subregion ∂R. Then,

$$S(\partial R) = \max_{\partial\partial R\subset\Sigma} \max_{v \text{ in } \Sigma} \int_{\mathcal{E}(\partial R)} n\cdot v \,, \tag{3.38}$$

where $\mathcal{E}(\partial R)$ is the union of Σ and the domain of dependence $D(R) = D^+(R)\cup D^-(R)$. Similar to the vector field v, where $\nabla\cdot v = 0$ on a Cauchy slice, generalize this to the full Lorentzian manifold by introducing a vector field V with same properties as v – therefore, $\nabla\cdot V = 0$. On $I^\pm(\partial\partial R)$, we will introduce some boundary conditions so that

[12] The origin of bit threads is from the sense of a max-flow-min-cut theorem in Riemannian geometry concerning the area-minimizing representative of some $m(\partial R)$ (the "bottleneck") and flow-maximizers. While we will use this in our discussion, the interested reader is directed to [42] for a detailed discussion on bit threads and this principle.

the role of Σ is replaced by a "smearing" field ϕ. Then, with the boundary conditions so that

$$\phi_{D^-(\partial\partial R)} = 0 \quad \text{and} \quad \phi|_{D^+(\partial\partial R)} = 1 \ ,$$

we have the bit threads formulation of HRT formula as [45]

$$S(\partial R) = \max_{V,\phi} \int_{D(\partial R)} n \cdot V \ . \tag{3.39}$$

Now one could ask what the bit threads bulk-boundary correspondence in AdS/CFT translates to in the case of de Sitter, if such could be described at all. In AdS/CFT, one motivates some nice things like subregion-subregion duality and entanglement wedge reconstruction, which are still not clear in the de Sitter context. Whether bit threads could make such a description is something not clear – however, a bit threads description of de Sitter is the first step towards looking at probable hints to such things.

3.2.1 Monolayer Proposal

In the monolayer proposal for bit threads in static patch, the bit threads originate from a horizon in a single-layered fashion. That is, they originate only towards one component and maintain the characteristic that they do not cross the horizon. In this sense, the usual bit threads formulation for the pode-antipode system is as follows: start by taking points on the horizons, "anchor" them so as to connect them via a codimension 1 surface. The bit threads then span the space between the anchor points. The bottleneck for this system would be at the horizon, giving us the Gibbons-Hawking entropy.

In the case of a Schwarzschild-de Sitter spacetime, this would has an interesting outlook [11]. There are two bottlenecks for the bit threads connecting the two horizons on the basis of where the bit threads are sourced, since if one considers a black hole at the center of one patch, an additional black hole is implied due to the entanglement in the TFD state, and in turn these are connected by an ER bridge.; the first bottleneck appears for the bit threads that escape the "bulge" of the pode (or antipode), and the second bottleneck appears for those bit threads connecting the two black holes in the wormhole.

3.2.2 Bilayer Proposal

In contrast to the monolayer proposal, where the bit threads are sourced solely towards one component, the bilayer proposal allows a horizon to source bit threads in a bilayered fashion, or in other words towards two components. Here, we consider the largest components as the only sources of bit threads, in the sense that cosmic horizons are the only factors we will consider. This can be imagined in static patch, for instance, by taking the horizon and source bit threads towards the pode (or antipode) and also in the usual sense described with the monolayer proposal. This has a very straightforward resolution, since the bit threads emitted towards the bulge end up at a bottleneck of vanishing area. In this case, there is no difference between the monolayer and the bilayer proposals, and gives the usual Gibbons-Hawking entropy.

However, for the example of a Schwarzschild-de Sitter spacetime, there is a very clear difference from the monolayer proposal. Again – since we will not consider bit threads from black hole horizons, the bit thread contributions will be solely from the cosmological horizons. Then, there are two sets of bit threads: first, which head to the cosmic horizon, and the other, which thread the wormhole connecting the two black holes. In this sense, the second layer of the cosmic horizon sourcing the second set of bit threads replaces the bit threads emitted from the black hole horizon in the monolayer proposal. This is equivalent to the monolayer proposal in the sense that the conditional entanglement entropy (given by the sum of the contributions from the cosmic horizon and the black hole horizon) remains the same under computation from both the proposals.

3.2.3 Semiclassical Limit

When including semiclassical corrections, we have to keep in mind that one has to extremize the generalized entropy. Following the static patch configuration of de Sitter space, we will study the monolayer and the bilayer proposals taking into account of these semiclassical corrections, to define a de Sitter HRT prescription.

Monolayer proposal: From the previous discussion, it is evident that the Cauchy slice on which we define anchor points has three slicings: a left one, corresponding to the pode side, an exterior one bounded by the anchor points, and a right slicing, corresponding to the antipode side. For the sake of convenience, for $\Sigma = \Sigma_{\text{left}} \cup \Sigma_{\text{ext}} \cup \Sigma_{\text{right}}$, we will denote by Σ_i one of these three components. If one had another spacelike slice Σ' with the same anchor points, the causal domains would coalesce, implying that the slices would be equivalent. Let the subregion $\mathcal{S}$ in $H_1 \cap H_2$ be in consideration – then, find a minimal extremal surface $\mathcal{X}_{\text{ext}}$ that is homologous to $\mathcal{S}$ on Σ'_{ext}. Then, the area term becomes[13] $S(\mathcal{S}) = \min_{\mathcal{X}_{\text{ext}}} \frac{\text{Area of } \mathcal{X}_{\text{ext}}}{4G_N\hbar}$. Taking bulk corrections into account, we would have:

$$S_{\text{gen}}(\mathcal{X}_{\text{ext}}) = \frac{\text{Area of } \mathcal{X}_{\text{ext}}}{4G_N\hbar} + S_{\text{bulk}} \,. \tag{3.40}$$

From this, the monolayer proposal in semiclassical limit yields the following:

$$S(\mathcal{S}) = \min\ \text{ext} \frac{\text{Area of } \mathcal{X}_{\text{ext}}}{4G_N\hbar} + S_{\text{bulk}} \,. \tag{3.41}$$

Bilayer proposal: In the bilayer proposal, the extremal surface whose area gives the generalized entropy is replaced by the sum of the areas of such surfaces in the three

[13] Here, we have restored the $\hbar$ as opposed to the previous discussion in bit threads to incorporate the leading order $G\hbar$ contribution, which is what the generalized entropy formula will contain. For this reason, we have restored $\hbar$, although this makes an appearance implicitly throughout these discussions.

components of Σ_i. The same approach is considered upto the extremal surface finding step. Here, we instead look for some $\mathcal{X}_{\text{left}}$ and $\mathcal{X}_{\text{right}}$ surfaces that are homologous to the corresponding subregions $\mathcal{S}_{\text{left}} = \mathcal{S} \cap H_1$ and $\mathcal{S}_{\text{right}} = \mathcal{S} \cap H_2$ respectively. However, there is a subtlety in the bulk corrections term – along with the additions of the homologous surfaces $\mathcal{X}_{\text{left}}$, $\mathcal{X}_{\text{ext}}$ and $\mathcal{X}_{\text{right}}$, we add the individual bulk slice contributions rather than simply the sum, as argued in [46]. Due to this, $S(\mathcal{S})$ in the bilayer fashion becomes:

$$S(\mathcal{S}) = \min\ \text{ext}\left[\sum_i \frac{\text{Area of } \mathcal{X}_i}{4G\hbar} + S_{\text{bulk}}\left(\bigcup_i \bar{\Sigma}_i\right)\right] , \tag{3.42}$$

where $\bar{\Sigma}_i$ corresponds to the slices adding to bulk corrections with the union taken.

4 Quantum de Sitter Spacetime

4.1 The Hilbert Space

To construct Hilbert space of de Sitter, we need to realize what are the (quantum) solutions of Wheeler-de Witt (WDW) equations for some asymptotically de Sitter with boundary conditions. Progress in this direction in AdS can be found in different contexts in [25,47]. The idea of Hilbert space being finite-dimensional is unique to de Sitter, as in flat-spacetime or AdS black hole, the Hilbert space is infinite-dimensional. One can argue that because of the phase space of gravitational solutions for not-so-precise boundary conditions in asymptotically de Sitter is compact. This results in the finite-dimensional Hilbert space [48]. We will review in this section the Hilbert space constructed for quantum gravity in de Sitter [8] and find that it is indeed finite-dimensional with a norm given by modified Higuchi's norm [49, 50]. As a side-line, we will also comment on the algebraic analysis of the algebra of observables for a time-like observer in semiclassical de Sitter static-patch and we will see that it agrees with the finite dimensionality of Hilbert space. The algebra is found to be of von Neumann algebra type II_1 factor [10,51]. While the von Neumann algebra for the black hole horizon is given by type II_∞. For reviews on von Neumann Algebra see [51–55]. Factor type III_1 is the emergent von Neumann algebra that describes the effective bulk at large N [56, 57] and type II_∞ describes the perturbations in $1/N$ corrections [58]. More appropriately, the operator algebra outside the black hole horizon in eternal Anti-de Sitter [3], dual to TFD, should be of type III_1 and for the $1/N$ corrections, which implies G_N corrections in the bulk, the algebra outside the horizon should be of type II_∞ which is less-strict in comparison to type III algebras since even if one can not define the microstates, one can define entropy for type II_∞ operator algebra but only for states which are associated with a trace class.

In this section, we will talk about two striking results that were formulated recently by Suvrat et al in [8] on the Hilbert space of quantum gravity in de Sitter in the sense of asymptotic quantization and by Witten et al (referred to as the CLPW proposal) in [10] on the algebra of observables in de Sitter in the framework of static patch. Before we start discussing these, we will quickly point out some aspects of the Hilbert space in quantum gravity that will be relevant for our discussion below in order:

1. *Perturbative Hilbert space:* Taking $|0\rangle$ to be the vacuum state, from perturbation theory one can construct states "on top" of $|0\rangle$ by introducing a function ψ and building with the field $\Phi(x_1 \dots x_n)$, as follows:

$$|\Psi\rangle = \int dx_1 \dots dx_n \psi_{(x_1)\dots\psi(x_n)}\, \Phi(x_1)\dots\Phi(x_n)|0\rangle\ . \tag{4.1}$$

   We note, however, that for $n > 0$, this usually corresponds to (1) states that are not invariant under de Sitter isometry group (except $|0\rangle$ itself), and (2) the fact that the Hilbert space of this theory is infinite-dimensional when n is sufficiently large. As for the former, we have to ensure that states found from solving the Wheeler-DeWitt equation satisfy the de Sitter isometry invariance constraint. As of the latter, as we shall see in subsection 4.1.1, we cannot precisely formulate the finite-dimensionality nature of the Hilbert space in the sense of perturbation theory with asymptotic quantization[14]. A nonperturbative description might pave way towards describing the finite-dimensionality of the Hilbert space, although this is something that has not been done.

2. *Invariance under de Sitter isometries:* As stated above, the states Ψ satisfying the Wheeler-DeWitt equation must be invariant under de Sitter isometries. However, other than $|0\rangle$, no other states satisfy normalizability or invariance conditions. In such cases, one can define a modified norm by dividing the norm $\langle\Psi, \Psi\rangle$ by the volume of the group vol(Diff $\times$ Weyl), as suggested by [49, 50, 59] to get a finite norm. In the perspective of refined algebraic quantization, this is rather intrinsically defined.

3. *CFT-like functionals:* In general, when doing asymptotic quantization, Wheeler-DeWitt states tend to take the form of

$$\lim_{\mathrm{vol}[g]\to\infty} \Psi[g, \Phi] \sim e^{\pm S[g,\Phi]} \mathcal{Z}_\pm[g, \Phi]\ ,$$

   where $S[g, \Phi]$ is a universal factor[15] (i.e. applies to all Wheeler-DeWitt states) built of some matter and gravitational terms that scale according to the asymptotics, and $\mathcal{Z}[g, \Phi]$ is a CFT partition-like quantity. That is, it satisfies the usual Diff $\times$ Weyl properties upto anomaly $\mathcal{A}_D$. The two signatures arise due to the Hamiltonian

[14] In finite-time "bulk" physics, we have to take into account of corrections to the interaction Hamiltonian, which in the late-time limits is not relevant in the sense of corrections arising from asymptotic-to-bulk limits.

[15] These can be related to holographic renormalization, in the sense of counterterms. In fact, in Cauchy slice holography this is how we can motivate the notion of deformations and counterterms, using which we construct holography from Cauchy slices, where the state Ψ lives, in the dual description of $Z[g, \Phi] \longrightarrow Z^{\Sigma}[g, \Phi]$, where $Z^{\Sigma}[g, \Phi]$ is a deformed partition function

constraint being of second-order, although usually one of the terms is considered *restricted*[16]. In fact, as we shall review, the functional $\mathcal{Z}[g,\Phi]$ is built of coefficient functions that behave similarly to correlators in the theory. However, these are not fully constrained from the Wheeler-DeWitt equation, due to which it would be incorrect to assume that since $\Psi[g,\Phi] \sim \mathcal{Z}[g,\Phi]$, both $\Psi[g,\Phi]$ and $\mathcal{Z}[g,\Phi]$ are fixed *uniquely.*

4.1.1 de Sitter Quantum Gravity Hilbert Space

In this subsection, we will discuss the Hilbert space of de Sitter quantum gravity as outlined by [8] in the context of asymptotic quantization. We will start by briefly reviewing their result. Before we start, it is important to start by remarking on the importance of asymptotic quantization schemes. The way to see this is as follows: fields for which we calculate n-point correlators in the bulk, $\langle \Phi(x_1)\dots\Phi(x_n)\rangle$ are dual to operators $\mathcal{O}_\Phi$ on the boundary in the following sense:

$$\langle \Phi(x_1)\dots\Phi(x_n)\rangle_{\text{bulk}} \sim \langle \mathcal{O}_\Phi(x_1)\dots\mathcal{O}_\Phi(x_n)\rangle_{\text{bdy CFT}} \,.$$

In a theory of holography between a bulk theory and a boundary theory, the operators live on a boundary CFT, and the generating functional $Z[g,\Phi]$ lives on the boundary. This way, the holographic dictionary is of the form of $\Psi[g,\Phi] \sim Z[g,\Phi]$. In an asymptotic quantum gravity theory, the general format is of the following form:

$$\lim_{\Gamma\to 0} \Psi[\bar{g},\bar{\Phi}] \sim \mathcal{Z}^{\pm}[\bar{g},\bar{\Phi}] \,, \tag{4.2}$$

where $\bar{g} = \frac{g}{\Gamma^2}$ is a rescaled metric, $\bar{\Phi}$ denotes the corresponding matter fields and $\mathcal{Z}^{\pm}$ are partition-like functions, whose meaning will be more apparent later, along with the distinction between Z and $\mathcal{Z}$. As of now, it is important to note that in the framework of asymptotic quantum gravity where $\Gamma \to 0$, the Wheeler-DeWitt states take the form of

$$\Psi[\bar{g},\bar{\Phi}] \sim e^{\text{CT}}\mathcal{Z}[\bar{g},\bar{\Phi}] \,. \tag{4.3}$$

Here, e^{CT} are counterterms, which we will make use of in our discussion on Cauchy slice holography, and comment on in the aftermath of the Hilbert space discussion in section 4.1. In the usual holographic dictionary, Z is the CFT generating functional for correlators – in the asymptotic dictionary (4.3), this has to do with insertions on an asymptotic slice Σ_{late}, where "late" corresponds to the asymptotic time slices in the de Sitter case[17].

[16] In the case of the de Sitter Cauchy slice holography analysis by [60], both the branches have to be considered; this will later be described in the Cauchy slice holography discussion in section 4.2.3.

[17] When working with deformations, this asymptotic nature is more explicit; in the AdS scenario, deformations to the AdS boundary parametrized by some λ (which essentially plays the same role as Γ above, with a slightly different meaning for the sake of Cauchy slice holography) would be radial slices, whereas in de Sitter this would be from $\mathcal{I}^{\pm}$ to finite time slices.

In the framework of canonical quantum gravity, we have the Hamiltonian and momentum constraint equations that are used to define a Wheeler-DeWitt state[18] derived from the decomposition of the metric into the lapse and shift functions. These constraints in the presence of a massive scalar contribution are:

$$\mathcal{H}\Psi[g,\Phi] = 0 \quad \text{Hamiltonian constraint} \ , \tag{4.4}$$

$$\mathcal{D}_a\Psi[g,\Phi] = 0 \quad \text{Momentum constraint} \ , \tag{4.5}$$

where the Hamiltonian constraint is referred to as the Wheeler-DeWitt equation. This can be expanded in the case of a non-zero Λ as

$$\left[\frac{16\pi G_N}{\sqrt{g}}\left(\Pi_{ab}\Pi^{ab} - \frac{\Pi^2}{D-1}\right) - \frac{\sqrt{g}}{16\pi G_N}\left(\mathcal{R} - 2\Lambda\right) + \text{matter and int. terms}\right]\Psi[g,\Phi] = 0 \ , \tag{4.6}$$

where $\Pi \equiv g_{ab}\pi^{ab}$ with π^{ab} the conjugate momentum,

$$\pi^{ab} = -i\frac{\delta}{\delta g_{ab}} \ . \tag{4.7}$$

In our interest, we seek for solving the WDW equation when the Λ term is dominant – in some sense, one sees that this seems equivalent to large-volume slices. In particular, the asymptotic limit $\text{vol}[g] \to \infty$ has been previously considered [see for instance in the case of WDW equation in AdS and recently in the context of Cauchy slice holography], since this can be seen to provide an intrinsic notion of time. Since the WDW equation does not depend on time variables, one can instead rewrite in terms of such "late-time" slices to solve the WDW equation. In the view of the cosmic No Hair theorem, the idea is that for those spacetimes that evolve into dominant Λ slices, such late-time slices can be used to find solutions to the WDW equation in terms of late-time physics. Therefore, we introduce a change of variables,

$$g_{ab} = \Omega^2\gamma_{ab} \ , \tag{4.8}$$

$$\Omega = |g|^{\frac{1}{2D}} \longrightarrow +\infty \Longrightarrow \text{logvol}[g] \to +\infty \tag{4.9}$$

from which one sees that large-volume slices are late-time slices. In this setting, it is now necessary to identify the effect of dominant Λ on matter fields, which can be identified in terms of the dilution variable O defined by $O = |g|^{\frac{\Delta}{2D}}\Phi$, where Δ is found by solving the WDW equation and $|g| \equiv \det(g)$. Then, one can rewrite the original Hamiltonian constraint (4.6) in terms of the intermediate variables γ_{ab}, Ω and O. The next step is to express the wavefunctional $\Psi[g]$ in the exponential $e^{i\mathcal{F}}$, where $\mathcal{F}$ is the sum of functionals X_i and Y_j (plus $\frac{1}{\Omega}$ terms), which correspond to the gravitational part and matter parts of the expansion in the limit $\Omega \to \infty$. One sees that the asymptotic solutions to the

[18] In what follows, the satisfaction of the momentum constraint by a WDW state will be implicit and will not be mentioned explicitly unless in cases where the diff-invariance properties of the $\mathcal{Z}[g,\Phi]$ functional is being discussed.

WDW equation take the following form, where $S[g,\Phi]$ is a universal factor appearing for all states:

$$\Psi[g,\Phi] = e^{iS[g,\Phi]}\mathcal{Z}[g,\Phi] \; , \tag{4.10}$$

where $S[g,\Phi]$ is built of summation of terms of the form X_{D-n} and $Y_{\beta-m}$. In the gravitational part of solving the WDW equation, one is concerned with determining terms of the form

$$\Omega\frac{\delta}{\delta\Omega}X_{D-2n} \tag{4.11}$$

The term $\mathcal{Z}[g,\Phi] = e^{iX_0}$ is then seen to have the following properties: first, it has the Weyl property

$$\Omega\frac{\delta}{\delta\Omega}Z[g,\Phi] = \mathcal{A}_D\mathcal{Z}[g,\Phi] \; , \tag{4.12}$$

where for even D, $\mathcal{A}_D$ can be computed explicitly. Second, in general, one can see that $\mathcal{Z}[g,\Phi]$ has a Diff $\times$ Weyl symmetry under simultaneous transformations in g and Φ. For instance, by doing the transformation $[g,\Phi] \to [g+\lambda\mathcal{L}_\xi g, \Phi + \lambda\mathcal{L}_\xi\Phi]$, one obtains diff-invariance. Similarly, Weyl invariance is found by a conformal transformation between g and the corresponding weight for Φ. One sees that the functional $\mathcal{Z}[g,\Phi]$ obeys diff-invariance and the Weyl transformation rule given by

$$\left(2g_{ab}\frac{\delta}{\delta g_{ab}} - \Delta\Phi\frac{\delta}{\delta\Phi}\right)\mathcal{Z}[g,\Phi] = \mathcal{A}_D\mathcal{Z}[g,\Phi] \; , \tag{4.13}$$

where we use the fact that one can calculate the anomaly $\mathcal{A}_D$ as follows: in the case of odd D, the anomaly $\mathcal{A}_D$ vanishes, whereas for even D, one can calculate the anomaly explicitly in terms of the curvature invariants and Euler densities associated to the anomaly. This can be expressed as:

$$\mathcal{A}_D \sim \frac{\sqrt{g}}{2\kappa^2}\left(\mathcal{E}_D + \mathcal{W}_D\right) \; , \tag{4.14}$$

where $\mathcal{E}_D$ is proportional to the D-dimensional Euler density and $\mathcal{W}_D \equiv W_{abcd}W^{abcd}$. This is up to $D=4$, while in greater than $D=6$ dimensions one requires additional terms into consideration. One can then see that evaluating the equation (4.13) is analogous to the trace anomaly equation for an even D CFT. For example, in $D=2$, (4.13) reduces to

$$\left(2g_{ab}\frac{\delta}{\delta g_{ab}} - \Delta\Phi\frac{\delta}{\delta\Phi}\right)\mathcal{Z}[g,\Phi] = \mathcal{A}_2\mathcal{Z}[g,\Phi] \; , \quad \mathcal{A}_2 = \frac{1}{16\pi G_N}i\sqrt{g}\mathcal{R} \; . \tag{4.15}$$

In this case, the analogy to the Brown-Henneaux central charge for a $D=2$ CFT can be found. Similarly, for $D=4$, one has the following anomaly term from the corresponding $\Omega\frac{\delta}{\delta\Omega}X_{D-4}$ expansion by considering $D=4$:

$$\mathcal{A}_4 = \frac{-i\sqrt{g}}{8\kappa^2}\left(R_{ij}R^{ij} - \frac{1}{3}R^2\right) \; . \tag{4.16}$$

This calculation is interesting for a reason; up to the factor of $-i$, (4.16) is analogous to that of the trace anomaly for a 4D CFT arising from holographic renormalization calculations [61]. From the standard form of the Weyl anomaly [62] with coefficients a and c for the A and type B anomalies respectively to be determine, we have

$$\frac{\sqrt{g}}{16\pi^2}\left(-a\mathcal{E}_4 + cW_{ijkl}W^{ijkl}\right) = \frac{-i\sqrt{g}}{64\pi G}\left(R_{ij}R^{ij} - \frac{1}{3}R^2\right) \tag{4.17}$$

$$= -a\left(R_{ijkl}R^{ijkl} - 4R_{ij}R^{ij} + R^2\right) + c\left(R_{ijkl}R^{ijkl} - 2R_{ij}R^{ij} + \frac{1}{3}R^2\right) \tag{4.18}$$

$$= \frac{-i}{4}\left(R_{ij}R^{ij} - \frac{1}{3}R^2\right) . \tag{4.19}$$

From this, we find that

$$a = c = \frac{-i\pi}{8G_N} . \tag{4.20}$$

This shows that

We can now be interested in the exact nature of $\mathcal{Z}[g,\Phi]$, which we can understand by rewriting it in terms of some multilinear functionals $\mathcal{G}_{n,m}$, formed by a collection of n tensor fields and m scalar field contributions:

$$\mathcal{Z}[g,\Phi] = \exp\left(\sum_{n,m}\kappa^n\mathcal{G}_{n,m}\right) , \tag{4.21}$$

and in the complete set of asymptotic WDW states expansion, we have

$$\Psi[g,\Phi] = e^{iS[g,\Phi]}\sum_{n,m}\kappa^n\delta\mathcal{G}_{n,m}\mathcal{Z}_0[g,\Phi] . \tag{4.22}$$

The functional $\mathcal{G}_{n,m}$ is combined of metric fluctuations and matter fluctuations defined in the near-flat metric perturbation $g_{ab} = \delta_{ab} + \kappa h_{ab}$, built of tensor fields (labelled by n) and scalar fields (labelled by m) in the usual convention as [8]:

$$\mathcal{G}_{n,m} = \frac{1}{n!m!}\int dy_1\ldots dy_n dz_1\ldots dz_m h_{i_1j_1}\ldots h_{i_nj_n}\Phi(z_1)\ldots\Phi(z_m)G_{n,m}^{y_1\ldots y_n,z_1\ldots z_m} . \tag{4.23}$$

While all the intrinsic nature of $\mathcal{G}_{n,m}$ will not be explicitly discussed in our review, we will only state that the set of coefficients $G_{n,m}^{\vec{i}\,\vec{j}}$ (where the arrow on top represents the set of indices running in n and m), which behave similarly to that of CFT correlators since the obey the same identities. Again, this similarity may only be upto some level where we are only concerned with the physical nature of $\mathcal{Z}[g,\Phi]$ or $G_{n,m}^{\vec{i}\,\vec{j}}$ obeying the Ward identities similarly to that of correlators. We will now remark on the CFT-like counterparts of these quantities and their deterministic structure on asymptotic quantization grounds.

In the scheme discussed above, the functional $\mathcal{Z}[g,\Phi]$ is *not* fully fixed – this can be seen by noting that one could add in terms that are Weyl-invariant to (4.13) and generate a different solution without explicit dependence on Ω. In order to understand the nature of the fixing of $\mathcal{Z}[g,\Phi]$, we need to understand in detail how the Hilbert space is being found in terms of the theory space, determined by the coefficient functions $G_{n,m}^{\vec{i}\,\vec{j}}$. One can start by writing the $\mathcal{Z}[g,\Phi]$ functional in the form given by (4.21). By expanding in terms of the Diff $\times$ Weyl transformations that $\mathcal{Z}[g,\Phi]$ obeys, where one can write the expansion in terms of h_{ab}. What we are doing here is to consider the perturbed case of g_{ab} by using the fact that in even dimensions, one can use a Weyl transformation (preserving topology, which is $\mathbb{S}^D$) to go from this perturbed setup to the regime where we have $g_{ab}^{\text{phys}} \sim \omega\,(\delta_{ab} + \kappa h_{ab})$, where ω is a Weyl factor. Similarly, the matter fields also receive a correction by some factor $\Phi \sim \omega^{\Delta}\Phi^{\text{phys}}$. Under the Diff$\times$Weyl transformation in the $\mathcal{Z}[g,\Phi]$ expansion, one sees that the coefficient functions $G_{n,m}^{\vec{i}\,\vec{j}}$ obey a set of Ward identities (a *trace* identity, obtained from the Weyl transformation and a *divergence* identity, obtained from diffeomorphism invariance), with the anomaly term in the perturbed regime also having the same vanishing property for odd D. These functions behave somewhat similarly as that of correlators $\langle T(x_1)...T(x_n), \Phi(z_1)...\Phi(z_n)\rangle$, defined for some operators T with spin-2 and Φ of spin-0. These identities prescribe the coefficient functions $G_{n,m}^{\vec{i}\,\vec{j}}$, and one can see that this would in turn, uniquely define WDW states. This is to say that $G_{n,m}^{\vec{i}\,\vec{j}}$ terms define a theory space, which uniquely find solutions to the WDW equation. Indeed, this is up to Weyl invariance; one can add an Ω-independent term and specify a different solution state.

However, we have not made any statements about the invariance of WDW states under de Sitter isometries. In the weakly-coupled case of de Sitter quantum gravity, Higuchi [49, 50] showed that the $\kappa \to 0$ limit has a Fock space, such that the states are invariant under SO$(1, D+1)$ and have normalized states, following Moncrief's conjecture. One can define a "seed state" $|\text{seed}\rangle$ by building on top of the Euclidean vacuum state $|0\rangle$ (this satisfies the de Sitter invariant group constraint, and is referred to as the Bunch-Davies state) for some set of scalar fields Φ_n; however, except $|0\rangle$ no states can be found that are invariant under such isometries. Further, these states are not normalizable, and have an infinite norm. Moncrief's conjecture can be stated as follows: if one takes these states in de Sitter and divides the infinite norm with the volume of the de Sitter group, one obtains normalized states [59]. What we wish to do now is to show that in the $\kappa \to 0$ limit, the above construction for the Hilbert space reduces to Higuchi's proposal. The motivation to this is simply that when one turns on gravity minimally in the $\kappa \to 0$ limit, one has to construct a Hilbert space of small fluctuations, where one now has to find the states invariant under de Sitter group, where one does not have states satisfying this condition. Define the seed state as defined previously in (4.1) as

$$|\text{seed}\rangle = \int dx_1 \dots dx_n \psi_{x_1 \dots x_n} \Phi(x_1 \dots x_n)|0\rangle \;, \tag{4.24}$$

where ψ is some square-integrable function As stated above, no states other than the vacuum state $|0\rangle$ are invariant under the de Sitter group G_{dS} due to the presence of the function of compact support for $n > 0$. In this sense, what we are doing is to find the constrained Hilbert space $\mathcal{H}^{G_{dS}}$ from the Hilbert space $\mathcal{H}_{dS}$, so that states in $\mathcal{H}^{G_{dS}}$ are invariant in G_{dS}. Higuchi [49, 50] suggested, following [59] that the average of $|\text{seed}\rangle$ over

G_{dS},

$$|\Psi\rangle = \int_{G_{dS}} d\mathcal{G}\; U|\text{seed}\rangle \tag{4.25}$$

(where $d\mathcal{G}$ is the Haar measure on the G_{dS} group). From Higuchi [50], one defines the norm

$$(\Psi,\Psi) = \frac{1}{\text{vol}(G_{dS})}\langle\Psi,\Psi\rangle = \int d\mathcal{G}\langle \text{seed}|U|\text{seed}\rangle\ . \tag{4.26}$$

It is then left to show that under these isometries,, the states formed in the nongravitational limit satisfy invariance under G_{dS}. Under the isometries (i.e. involving translations, rotations, dilatations and SCTs), one sees that in the nongravitational regime, one has states invariant under G_{dS}. Later, in our review of holography of information [9], we will discuss some aspects of these isometries in the sense of residual gauge transformations.

4.1.2 Algebra of Observables in Static Patch

Here we will discuss the type II_1 von Neumann factor as the algebra of the observables of some static patch in de Sitter [10]. We can briefly summarize the relevance of type II_1 factors. Mathematically [54, 63], we can define a factor type II_1 $\mathcal{A}$ on $\mathcal{H}$ for which the range of the dimension function[19] $\mathcal{D}$ is in the interval $[0,1]$. For such a factor, we can define a function $tr\colon M \to \mathbb{C}$. A trace is called normalized if $tr(1) = 1$, type II_1 has this feature. That means the trace is defined for all the states associated with this algebra. An interesting system that it describes is the infinite qubits in system A being entangled with infinite qubits in system B [52]. However, due to infinite entanglement, this is infinite but we can renormalize it.

It is interesting to note that type II_1 factor is studied because of its complexity. While, in contrast, type II factors are easier with entropy than type III factors. This has to do with the UV divergent entropy associated with a density matrix in type III and there does not exist a trace at all for such factors. Any local quantum field theory region is of type III factors with divergent entropy, see [52] for more. However, one can define relative entropy between two states without any divergence. Factors (and hyperfinite factors) of type III were studied by Connes [64]. A simple quantum mechanics system, for example, harmonic oscillators, is described by the algebra of type I.

Our concern is to find the algebra of observables in de Sitter rather than a local region of QFT. It turned out that type II factor was the most appropriate for defining entropy and Hilbert space for de Sitter. Such type II von Neumann algebra does not permit an irreducible representation in Hilbert space. The same is true for type III von Neumann algebra.

[19] Dimension function is $D\colon \mathcal{P} \to \mathcal{R}$. Where $\mathcal{P}$ is an ordered set of projections and $\mathcal{R}$ is a range associated with each factor. For type II_1, $\mathcal{R} = [0,1]$ and for type II_∞, it is just the interval $[0,+\infty)$. Factors, except type III, were initially classified in this manner.

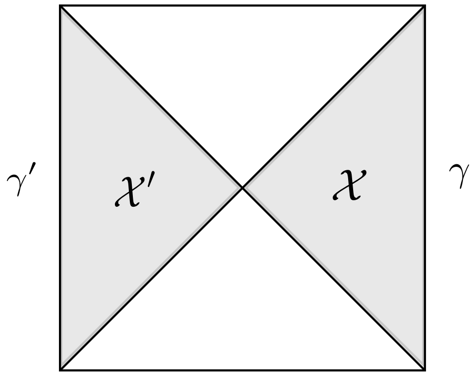

Figure 2: Observer γ can access only $\mathcal{X}$ while γ' can access only $\mathcal{X}'$. These two regions are spacelike separated and causally disconnected.

Let us recall some of the facts about von Neumann algebra. We define a region $\mathcal{O}$ for which there exists a Hilbert space $\mathcal{H}$. On $\mathcal{H}$, the collection of operators $B(\mathcal{H})$, which act on $\mathcal{H}$, define an algebra $\mathcal{A}$. This algebra $\mathcal{A}$ is defined such that the projection p of its commutant[20] $\mathcal{A}'$ defines $p\mathcal{H}$ which is a subspace of $\mathcal{H}$. We can conclude that the full algebra of operators of $B(\mathcal{H})$ is given by $\mathcal{A}$ and its commutant $\mathcal{A}'$. For $dim(\mathcal{H}) = n < \infty$, there is a bi-commutant theorem $\mathcal{A} = A''$. That means that every element a that exists in $\mathcal{A}$ also exists in $\mathcal{A}''$. A von Neumann algebra is called *factor* if its center is $\mathbb{C}\mathbf{1}$. In other words, $Z = A \cap A' = \mathbb{C}\mathbf{1}$. Any von Neumann algebra $\mathcal{A}$ is made up of these factors. These factors are type I, type II, and type III. Naively, the classification can be done by studying the dimension function $\mathcal{D}$. Every factor has its definition of trace. For instance, trace does not exist in type III algebra which naively speaking trace is defined for type I. For type II_1, a trace is defined (and normalized $tr1 = 1$) for every density matrix ρ. While type II_∞ admits trace to only those states belonging to a class called 'trace class'. A mathematically oriented discussion is available in [54, 63, 65] and a physics-oriented discussion is given in [51–53, 55].

In any way, type II_1 admits a trace and thus we can define the entanglement entropy in the region $\mathcal{O}$. However, this entropy needs to be renormalized. We always assume that the entropy we write in type II_1 is a renormalized entropy, up to an additive constant. As a side comment, there is no way to define the trace because there is a UV divergence in a density matrix in type III algebra, thus there is no notion of entropy [66, 67]. Instead, we can define 'relative entropy' [52]. This is where Tomita-Takesaki theory becomes most useful. Thus if we have a QFT fixed in de Sitter background, without gravity, then naively the algebra should be of type III. In detail, if the right-static patch for some worldline observer γ is $\mathcal{X}$, then the algebra $\mathcal{A}$ on $\mathcal{H}$ would be of type III. Moreover, $\mathcal{X}'$ is defined to be the left-static patch for an observer γ'. It would be misleading to place γ and γ' on the left and right boundary. As we have discussed, these are not boundaries of de Sitter. Algebra on $\mathcal{X}$ is $\mathcal{A}$ and algebra on $\mathcal{X}'$ is $\mathcal{A}'$ where $\mathcal{A}$ and $\mathcal{A}'$ are commutants

[20] Every $a \in \mathcal{A}$ and every $b \in \mathcal{A}'$ commutes, i.e, $[a, b] = 0$.

and define an algebra type III. The interesting differences appear only when we include gravity in the picture.

It is interesting to observe that type II_1 algebra has a state with maximum entropy, which is defined to be zero in this algebraic approach, and any state must have lower entropy than zero. We will see how exactly the empty de Sitter vacuum is that maximum entropy state [68]. This all will be done for a worldline observer in a static patch. There is not a clear complete algebra for the global de Sitter spacetime.

Since we wish to write the algebra for γ, one could establish that the same algebra type is found for the whole static patch $\mathcal{X}$. This is proven by the use of 'timelike tube' theorem [69, 70]. Conceptually, this theorem asserts that for a woldline γ we can make an envelope around γ which would consist of all the points which can be reached by deforming $\gamma \in \mathcal{O}$. We will call such an envelope as timelike envelope $\mathcal{E}$. In real analytic spacetime, timelike tube theorem says

$$A(\gamma) = A(\mathcal{E}(\gamma)) \tag{4.27}$$

and one can say that $\mathcal{X}$ lies in this envelope $\mathcal{E}$. For a simpler analogy, this sort of equivalence can also be said about domain of dependence (spacelike tube) of some Cauchy slice Σ. That would mean that algebra for $\Sigma \in \mathcal{O}$ is equal to the algebra of the domain of dependence $D(\Sigma)$. This is simple for ordinary quantum field theory with a fixed background. For recent discussions on timelike tube theorem in curved spacetime see [71, 72].

Now we will turn on the gravity by setting $G_N \to 0$. The effect of turning on the gravity would be minimal and can be operated approximately in the leading order in perturbation theory. There does not exist a non-perturbative way to analyze such problems. In such an approximation, we can quantize gravity for free fields in de Sitter. This creates a Hilbert space $\mathcal{H}_G$ which represents the fluctuations in the gravity. Now we also quantize the matter fields, this can be represented by $\mathcal{H}_M$. We can write a tensor product of two Hilbert space

$$\mathcal{H} = \mathcal{H}_G \otimes \mathcal{H}_M \tag{4.28}$$

where $\mathcal{H}$ is the Hilbert space we started with.

We now must define what are the states which are invariant under de Sitter isometries group $G_{dS} = SO(D, 1)$. These would be the normalizable state in the Euclidean vacuum called 'Bunch-Davies' state Ψ_{dS} (normalized in the sense $\langle \Psi_{\mathrm{dS}} | \Psi_{\mathrm{dS}} \rangle = 1$). We have discussed these isometries in subsection 2.2. Just like Hartle-Hawking state in a black hole, these states are supposed to have a thermal interpretation [4, 73] in the form of correlation function in state Ψ_{dS}, which has an importance in our discussion. An algebraic argument about the thermal interpretation in Ψ_{dS} states is that the modular Hamiltonian which generates the automorphism group of the algebra in the static patch is given by

$$H_{\mathrm{mod}} = \beta_{\mathrm{dS}} H \tag{4.29}$$

where H is the Hamiltonian of de Sitter. A similar interpretation in the context of black holes for similar Hartle-Hawking state can be found in [74].

Our Hilbert space $\mathcal{H}$ in eq. (4.28) will contain Ψ_{dS} which would be analytically continued from Euclidean sphere S^D. It was found that Ψ_{dS} has a flat entanglement spectrum [75]. The most important fact about Ψ_{dS} is that it has a maximal entropy that is de Sitter invariant and every other state has entropy than $S_{\rho_{dS}}$. Sometimes it is useful[21] to calculate the relative entropy of any state Φ with Ψ_{dS}

$$S(\sigma|\rho_{dS}) = \text{Tr } \sigma(\log\, \sigma) - \text{Tr } \sigma(\log\, \rho_{dS}) \tag{4.30}$$

where ρ_{dS} is the density matrix for state Ψ_{dS} and σ is the density matrix for state Φ. Of course, we can associate density matrices to states only when the algebra is type I or type II, otherwise for type III, we use a more abstract setting of relative entropy in terms of modular operators. When $\rho_{dS} = 1$, which is the maximal entropy state, we have

$$S(\sigma) = -S(\sigma|\rho_{\text{dS}}) \tag{4.31}$$

which is a way to interpret entropy as relative entropy. A similar approach was used in [76].

But the caveat about Hilbert space is that $\mathcal{H}$ is not the actual Hilbert space to understand the algebra. By turning the gravity, we must now look at a constrained Hilbert space $\hat{\mathcal{H}}$ [77]. These are gauge constraints imposed by the 'automorphisms' of de Sitter space but only in the static patch. For Euclidean de Sitter, this is the subgroup $G_P = \mathbb{R}\times SO(D-1)$, where $\mathbb{R}_t$ is generated by Hamiltonian H of de Sitter and $SO(D-1)$ is simply the rotation group of the sphere. One can see this constraint as 'Gauss Law' of gravity which exists even for small G_N. But no state, other than vacuum state Ψ_{dS}, is invariant under G_P. To find the states which are invariant under G_P in the static patch, we can do group averaging

$$\Psi = \int dU\ \Psi_{\text{dS}} \tag{4.32}$$

where dU is a Haar measure on G_P. These states are non-normalizable and uniquely considered for de Sitter by Higuchi [49, 50], see also [77] for discussions on the integral convergence and appendix of [10] for an interpretation in terms of BRST cohomology. We also have a proposed norm for these states Ψ

$$\langle \Psi, \Psi\rangle = \frac{1}{vol(G_P)} \int dU\ \Psi_{\text{dS}}. \tag{4.33}$$

Finding the algebra of observables: It is now given to us that operators would be from a constrained algebra $\hat{\mathcal{A}}$ that are imposed due to our model

$$\hat{H} = H_{\text{bulk}} + H_{\text{obs}} \tag{4.34}$$

[21] The relative entropy between states are always defined regardless of which algebra we use since the definition is independent of existence of a 'trace'. However, note that entropy is sensitive to what algebra is in consideration.

and the constrained algebra is the $\hat{H}$ invariant part of Hilbert space $\hat{\mathcal{H}}$

$$\hat{\mathcal{A}} = (\mathcal{A} \otimes B(L^2(\mathbb{R}_+))) \tag{4.35}$$

where $B(L^2(\mathbb{R}_+))$ is the algebra of the bounded operators acting on $L^2(\mathbb{R}_+)$ part of the Hilbert space $\hat{\mathcal{H}} = \mathcal{H} \otimes L^2(R_+)$. What this means is that any operator in $(\mathcal{A} \otimes B(L^2(\mathbb{R}_+)))$ that commutes with $\hat{H}$ is a part of $\hat{\mathcal{A}}$. This is technically a crossed product structure. Let us first some obvious operators in $(\mathcal{A} \otimes B(L^2(\mathbb{R})))$ that commutes with $\hat{\mathcal{H}}$. For a simple model where $H_{\text{obs}} = q$, the first relevant operator is $q = i\frac{d}{dp}$ itself. For that matter, for any element a $\in \mathcal{A}$, one can write $\{e^{ipH} \mathrm{a} e^{-ipH}\}$ which also commutes with $\mathcal{H}$. It is somewhat known that there should not exist more operators that commute with $\mathcal{H}$. This is also confirmed from Takesaki duality [10, 78], for a duality that exists between $(\mathcal{A} \otimes B(L^2(\mathbb{R})))$ and a double crossed product product.

Since the relevant operators $\{e^{ipH} \mathrm{a} e^{-ipH}, q\}$ look like forming a crossed product which would commute with $\mathcal{H}$, we now will write the relevant algebra as $\mathcal{A}_{\text{cr}}$. As per the reasons mentioned above, any operator B $\notin \mathcal{A}_{\text{cr}}$ does not take part in our algebra. This crossed product is actually between $\mathcal{A}$ and the one-parameter automorphism group generated by H [58]. What is this algebra? It is a typical type II_∞ algebra. We have not constrained H_{obs} until this moment.

Taking $H_{\text{obs}} \geq 0$ is bounding the observer's energy. To find states with H_{obs} constraint in $\mathcal{A}_{\text{cr}}$ we multiply it with Heaveside theta function

$$\Theta(q) = \begin{cases} 1, & q \geq 0 \\ 0, & q < 0 \end{cases} \tag{4.36}$$

which can be seen as mapping the Hilbert space $\Theta(\mathcal{H} \otimes L^2(\mathbb{R})) = (\mathcal{H} \otimes L^2(\mathbb{R}_+))$ and thus changing the von Neumann algebra from $\mathcal{A}_{\text{cr}}$ to $\hat{\mathcal{A}}_{\text{cr}}$. More formally, if one calls Θ as a projection operator Π, then

$$\hat{\mathcal{A}}_{\text{cr}} = \Pi \mathcal{A}_{\text{cr}} \Pi \tag{4.37}$$

which is a type II_1 algebra. While the type II_∞ does not admit states for any 'non-tracial' states, this $\hat{\mathcal{A}}_{\text{cr}}$ since it is a type II_1 algebra, admits a trace for every state. It can be also shown that $\hat{\mathcal{A}}_{\text{cr}}$ is a factor, since for $\Pi \mathcal{A}_{\text{cr}} \Pi$ and its commutant $\Pi \mathcal{A}'_{\text{cr}}$, the intersection only contains complex scalars. The commutant $\Pi \mathcal{A}'_{\text{cr}}$ where $\mathcal{A}'_{\text{cr}}$ is the commutant of $\mathcal{A}_{\text{cr}}$ in Hilbert space $(\mathcal{H} \otimes L^2(\mathbb{R}_+))$ [54].

Now we come back to our static patch. Since we have turned on gravity and included an observer[22] ($H_{\text{obs}} = q \geq 0$), we will define the maximum entropy state Ψ_{obs}

$$\Psi_{\text{obs}} = \Psi_{\text{ds}} e^{-\beta_{dS} q/2} \sqrt{\beta_{dS}} \tag{4.38}$$

where the correlations functions are defined in ψ_{dS} and the observer energy has a thermal distribution interpretation at the de Sitter temperature. The algebra for this collection of states is $\hat{\mathcal{A}}_{\text{cr}}$ discussed above. Since we had started with a normalized vacuum

[22] We have chosen q for H_{obs} since it is simple. We can chose any operator that commutes with H_{obs}.

$\langle\Psi_{\rm dS}|\Psi_{\rm dS}\rangle = 1$, we have

$$\langle\Psi_{\rm obs}|\Psi_{\rm obs}\rangle = 1. \tag{4.39}$$

We can argue that $\Psi_{\rm obs}$ is the maximum entropy state in de Sitter with gravity and observer in it. Traces for the states are given by factor type II_1. For example any $\mathrm{A} \in \Pi\mathcal{A}_{\rm cr}\Pi$, we can define

$$\mathrm{Tr}\ \mathrm{A} = \langle\Psi_{\rm obs}|\mathrm{A}|\Psi_{\rm obs}\rangle \tag{4.40}$$

and because of eqn. (4.39)

$$\mathrm{Tr}\ 1 = 1 \tag{4.41}$$

The Hilbert space, let us briefly call it $\mathcal{H}$, is generated by states $\mathrm{A}\Psi_{\rm obs}$. Since the maximally entropy state[23] has $\rho = 1$, the von Neumann entropy for such state

$$S(\Psi_{\rm obs}) = -\mathrm{Tr}(\rho\ \log\rho) \tag{4.42}$$

which is equivalent to

$$S(\Psi_{\rm obs}) = -\mathrm{Tr}(1\ \log 1) = 0 \tag{4.43}$$

Then by the virtue of generalized Second Law [68], any state $\Psi \in \mathcal{H}$ has the following property

$$S(\Psi) < S(\Psi_{\rm obs}) = 0. \tag{4.44}$$

Note that this is saying that any state we find other than $\Psi_{\rm obs}$ lies before $\Psi_{\rm obs}$ in the light cone.

We must also note that after gravitationally dressing our operators, it follows that for any operator $\mathrm{A} \in \hat{\mathcal{A}}_{\rm cr}$, the expectation value $\langle \mathrm{A}\rangle_{\rm obs}$ is equivalent to its expectation value in $\Psi_{\rm dS}$

$$\langle\Psi_{\rm obs}|\mathrm{A}|\Psi_{\rm obs}\rangle = \langle\Psi_{\rm obs}|e^{ipH}\mathrm{a}e^{-ipH}|\Psi_{\rm obs}\rangle = \langle\Psi_{\rm obs}|\mathrm{a}|\Psi_{\rm obs}\rangle \tag{4.45}$$

where $\mathrm{a} \in \mathcal{A}$. This is because $\Pi\Psi_{\rm obs} = \Psi_{\rm obs}$ and $H\Psi_{\rm obs} = 0$, followed with eqn. (4.39).

We now conclude the subsection with a few remarks in order.

1. An empty de Sitter without gravity is described by the type III algebra and the maximum entropy state is $\Psi_{\rm dS}$ with density matrix $\rho = 1$. After including gravity and an observer (by simply choosing $H_{\rm obs} = q \geq 0$), we impose some constraints on the states. Including gravity is followed by the automatic gravitational dressing of operators $\phi(\tau)$. Unlike the situation without gravity, $\phi(\tau)$ needs no smearing. The Hilbert space, after imposing gravity and observer energy constraint, is $\mathcal{H}_{\rm obs} = \mathcal{H} \otimes L^2(R_+)$. The algebra for the Hilbert space $\mathcal{H}_{\rm obs}$ is $\mathcal{A} \otimes B(L^2(R_+))$, where $\mathcal{A}$ is the algebra in Hilbert space $\mathcal{H}$. This is a manifestation of the crossed-product

[23] A maximally entropy state has density matrix ρ defined as multiples of the identity. In our case, it is exactly the identity.

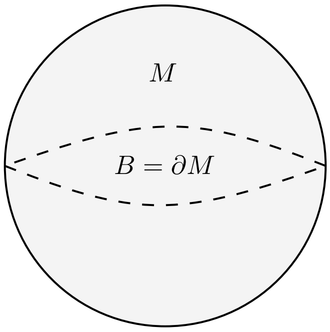

Figure 3: A sphere S^D, with hemisphere M and the boundary is D-1 sphere $B = \partial W$. On this, the quantum fields lie along B and are described by state Ψ_{dS}. In the path integral, we integrate over the manifold M and quantum fields ϕ_M.

algebra between type III algebra $\mathcal{A}$ and the one-parameter automorphism of the group (R_+) generated by H. The algebra can be written as

$$\hat{\mathcal{A}}_{\mathrm{cr}} = \Pi \mathcal{A}_{\mathrm{cr}} \Pi \tag{4.46}$$

where $\mathcal{A}_{\mathrm{cr}}$ is the crossed product algebra and Π is the projection operator $\Theta(q)$. The algebra, also a factor, $\hat{A}_{\mathrm{cr}}$ is a type II_1 algebra.

2. If we include gravity and an observer in the empty de Sitter, we get an extension of Ψ_{dS} to Ψ_{obs}

$$\Psi_{\mathrm{obs}} = \Psi_{\mathrm{ds}} e^{-\beta_{dS} q/2} \sqrt{\beta_{dS}} \tag{4.47}$$

where the observer energy has a thermal interpretation at the de sitter temperature. The quantum fields will be defined in Ψ_{dS}. Ψ_{obs} is still the maximum entropy state, of course, it can only make sense after including the observer. We find that the algebra for these states is of type II_1 since $\hat{A}_{\mathrm{obs}}$ defines a state with the maximum entropy, which is zero, see above for more discussion on this. But the important point to note is that maximum entropy was not found to be vanishing [4]. This means that entropy in type II_1 is being defined up to an additive constant. This additive constant is independent of the states. See [10] for the discussion of this additive constant. As a side comment, we add that the only perturbation we seek is $\mathcal{O}(1)$ not $\mathcal{O}(1/G)$ in the empty de Sitter.

3. At last, we want to remark on the similarities[24] between Ψ_{dS} and Hartle-Hawking state Ψ_{HH}. Originally, Hartle-Hawking state Ψ_{HH} was defined as the path-integral on a manifold M. It is called a no-boundary vacuum since there is no boundary

[24] Hartle-Hawking no boundary state Ψ_{HH} is an analog of Ψ_{dS} in gravity. A state defined in dS, as in sec. 4.1.1 as wave-functional, can be also defined as an analytical continuation of Hartle-Hawking 'wave-functional'. For a recent discussion of the algebraic analysis of Ψ_{dS} and Ψ_{HH} see [79]. It can also be dubbed as a 'universal' maximum entropy state for some spacetime, if one can define a trace $\langle \Psi_{\mathrm{HH}} | \mathrm{a} | \Psi_{\mathrm{HH}} \rangle$ and Ψ_{HH} is normalizable.

except just one boundary $B = \partial M$ on which the state is defined. Let us see what this makes for $\Psi_{\rm dS}$. $\Psi_{\rm dS}$ is defined for an Euclidean de Sitter which is a sphere S^D. Let us define a hemisphere M in this sphere and the boundary of this hemisphere is $B = \partial M$. $\Psi_{\rm dS}$ will define the quantum fields along this equator B. The path-integral on the manifold M gives, without the inclusion of the gravity

$$\Psi_{\rm dS} = \int_{\partial M|M} D\phi_M e^{-I(\phi_M)} \tag{4.48}$$

where ϕ_M is the quantum field in M. $\Psi_{\rm dS}$ would define quantum fields along ∂M. Here, we just summed up one possibility, namely ϕ_M, with the boundary condition ∂M. When we add the gravity, we need to sum over all the manifold with such boundary conditions and $\Psi_{\rm dS}$ will now also depend on the metric of M. Similarly, we define the Hartle-Hawking state. This means that $\Psi_{\rm HH}$ is a gravitational extension of $\Psi_{\rm dS}$. And since $\Psi_{\rm obs}$ is an observer extension of $\Psi_{\rm dS}$, one can hope to understand an observed added to $\Psi_{\rm HH}$. For an excellent discussion on this, see the recent paper [79]. The reasonable answer, in this context, is an algebra of type II. In [79], the author hopefully also introduces a background independent algebra at the cost of defining this algebra, not as Hilbert space algebra, but as operator algebras [80]. That is to say that we should be able to define an algebra $\mathcal{A}_{\rm ind}$ such that $\langle \Psi_{\rm HH}|{\rm a}|\Psi_{\rm HH}\rangle\,, {\rm a} \in \mathcal{A}_{\rm ind}$ can be defined without the knowledge of the spacetime. It is, however, subtle to know if there exists a maximum entropy state in 'any' spacetime. The existence of the maximum entropy state in some spacetime depends if the $\Psi_{\rm HH}$ is a normalizable state (which means that the trace is defined and the algebra is type II_1) or if $\Psi_{\rm HH}$ is rather a 'weight' and unnormalizable (which means that the trace is not defined and the algebra is type II_∞) [54]. There is no reason to believe that a maximum entropy state exists in a spacetime where $\Psi_{\rm HH}$ is an unnormalizable rather than a normalizable state [79].

4.2 de Sitter Holography

In AdS/CFT duality, we enjoy the correspondence between bulk states and boundary operators. Roughly that means if we add an operator in the worldline, it would correspond to a state being associated with the same operator. This is manifested strongly in the context of string theory. However, we do not need string theory to observe this duality. There have been many insightful developments in the past few decades in AdS/CFT (or Gauge/Gravity) duality. One of the most important ones might be holographic entanglement entropy.

A natural question to ask is what is the holographic dual of de Sitter? What are the holographic degrees of freedom in de Sitter? Is there any state-operator correspondence in de Sitter? Is there any asymptotic quantization in de Sitter? These are a few questions of importance. While de Sitter holography is not as straightforward as Anti-de Sitter holography, we do have some possible answers applicable to de Sitter, which open up new insights about geometry and entanglement.

4.2.1 Global dS/CFT

Here, we will describe the earliest proposal for de Sitter holography, where the boundaries are at $\mathcal{I}^{\pm}$. In this proposal, we start from a hemisphere in global de Sitter in coordinates bounded by a time $-\pi/2 \leq \tau 0$ in coordinates

$$ds^2 = R_{\text{dS}}^2 \left(d\tau^2 + \cos^2 \tau d\Omega_D^2\right) \ ,$$

. Starting from some $t = 0$ and moving to $t = t_\infty$, we pick

$$\Psi_{\text{dS}}[g_0, \Phi_0] = \int \mathcal{D}g\mathcal{D}\Phi \ e^{iS[g,\Phi]} \ ,$$

where Ψ_{dS} satisfies the WDW equation (4.6) and $g = g_0|_{t=\infty}$. The hemisphere then evolves into the full Lorentzian geometry, so that a boundary CFT at t_∞ describes correlators. That is, the two-point correlation function $\langle\Phi(x_1)\Phi(x_2)\rangle_\Psi$ can be computed usefully, by placing $|\Psi\rangle$ on a hemisphere with a boundary condition provided by Φ_0 as noted above. Then, the full sphere path integral would compute $|\Psi\rangle$ and $\langle\Psi|$. Then, the generating functional $Z[g, \phi]$ of these correlators is said to define the boundary dual of Ψ_{dS}:

$$\Psi_{\text{dS}}[g, \Phi] \sim Z[g, \Phi] \ . \tag{4.49}$$

While this looks strikingly similar to what we would expect from the usual quantum gravity dictionary, we must first look at whether or not this duality has a physical interpretation. The obvious first question is, "how are the operators on the boundary being dual to the bulk fields?" In AdS/CFT, this was rather convenient – CFT operators on the boundary $\mathcal{O}(x_n)$ were dual to bulk fields $\Phi(x_n)$. In our case, we can try the case of a massive scalar field. Before we do so, let us first discuss the settings of the correspondence.

The action for the overall theory can be written in terms of the bulk action contribution, which is the pure gravitational part of the action, and the Gibbons-Hawking-York (GHY) term, which expanded would look like

$$S = \frac{1}{16\pi G_N} \int d^D x \sqrt{-g} \left(R - \frac{2}{l_{\text{dS}}^2}\right) + \frac{1}{8\pi G_N} \int_\partial d^{D-1} x \sqrt{\gamma} \mathcal{K} \ , \tag{4.50}$$

where γ is the induced boundary metric and $\mathcal{K}$ is the trace of the extrinsic curvature. In the limit that t goes to ∞, this action is divergent; however, this can be made finite by adding local counterterms so that the action (4.50) is modified into $S \equiv S + S_{\text{counterterms}}$. The overall action then is

$$S = \frac{1}{16\pi G_N} \int d^D x \sqrt{-g} \left(R - \frac{2}{l_{\text{dS}}^2}\right) + \frac{1}{8\pi G_N} \int_\partial \sqrt{\gamma}\mathcal{K} + \frac{1}{8\pi G_N l_{\text{dS}}} \int_\partial \sqrt{\gamma} + \text{ matter terms} \ . \tag{4.51}$$

The matter terms are now irrelevant at the boundary $\mathcal{I}^-$ in consideration. The stress-tensor can be calculated in the standard field theoretic way by the variation $T^{\mu\nu} = -\frac{4\pi}{\sqrt{\gamma}}\frac{\delta S}{\delta\gamma_{\mu\nu}}$. This becomes [2]

$$T^{\mu\nu} = \frac{1}{4G_N}\left[K^{\mu\nu} - \gamma^{\mu\nu}\left(\mathcal{K} + \frac{1}{l_{\rm dS}}\right)\right] . \tag{4.52}$$

From this, one can deal with the stress tensor for asymptotically past (or future) de Sitter spacetimes at $\mathcal{I}^-$ (or $\mathcal{I}^+$) and find the central charge. However, what is the nature of the conformal field theory that is dual to the bulk field? To clear this, first, we will address the issue of *where* the CFT really lives.

In global coordinates, it seems initially that there is the issue of having two boundaries at $\mathcal{I}^\pm$ – which boundary are we dealing with when working with the state-operator correspondence? The answer is that there is only one dual Euclidean boundary. As argued by Strominger in his construction of dS/CFT, the Euclidean field theory lives only on a single sphere, rather than two, since a two-point correlator with x_1 on $\mathcal{I}^\pm$ and x_2 on $\mathcal{I}^\mp$ has the same form as a correlator with both points x_1 and x_2 on $\mathcal{I}^\pm$. Due to this, the boundary CFT is interpreted as living on a sphere, with the correlators of points on $\mathcal{I}^\pm$ identified in an antipodal way with respect to those on $\mathcal{I}^\mp$.

If we consider the field Φ in the setting of planar de Sitter space (2.9) , the field would be a function of η, and can be written at late times in the form of the *extrapolate* dictionary[25],

$$\Phi(\eta) \sim \eta^{\Delta_+}\mathcal{A}(x)\mathcal{O}^+_\Phi + \eta^{\Delta_-}\mathcal{B}(x)\mathcal{O}^-_\Phi , \tag{4.53}$$

where $h^\pm$ are the conformal weights and the terms $\mathcal{A}(x)$ and $\mathcal{B}(x)$ contain information about the operator that is dual to the field [81]. Then, the operator dual to this would have the conformal weight in $\pm$:

$$\Delta_\pm = \frac{1}{2}\left[(D-1)^2 \pm \sqrt{(D-1)^2 - 4(ml_{\rm dS})^2}\right] , \tag{4.54}$$

where we notice that the boundary conditions we wish to impose on $\mathcal{I}^\pm$. We can see that when the term $ml_{\rm dS}$, when sufficiently large, dominates over the $(D-1)^2$ term in (4.54) and the conformal weight becomes complex. Due to this, we can notice the non-unitary nature of the dual CFT we noted earlier in section 3. The final statement is that insertions x_i on $\mathcal{I}^\pm$ define correlators that have a dual operator $\mathcal{O}_\Phi$ in the CFT so that

$$\langle\Phi(x_1)\ldots\Phi_{x_n}\rangle_{\rm bulk} \sim \langle\mathcal{O}_\Phi(x_1)\ldots\mathcal{O}_\Phi(x_n)\rangle_{\rm CFT} , \tag{4.55}$$

where the dimensionality is of the form $\mathrm{dS}_D/\mathrm{CFT}_{D-1}$. As noted earlier, the boundary CFT lives on a single sphere $\mathbb{S}^{D-1}$.

[25] As opposed to AdS/CFT, there are two dictionaries that can be identified in this sense – the *differentiate* dictionary and the extrapolate dictionary. In the case of the former, one has only one conformal weight associated to a bulk operator. On the other hand, the extrapolate dictionary has two conformal weights conjugate to each other, $\Delta_\pm$, which indicates that in the picture of bulk reconstruction one has to take dual CFT operators at both the boundaries. See section 4.2.3 for a brief discussion on this.

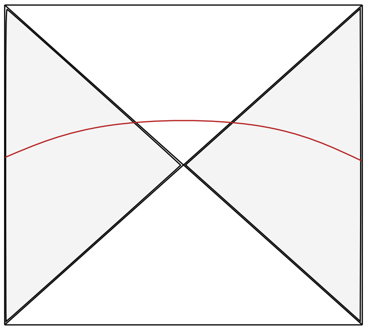

Figure 4: A spacelike slice at time t_1 is time independent. The topology of these spacelike slices is $S^{(D-1)}$ spheres.

4.2.2 Holography of Information

In section 4.1.1, we constructed a Hilbert space and checked with Higuchi's construction in the non-gravitational limit. Holography of information generally means that the information inside a shell is available on the boundary. It does not require a holographic definition to describe the degrees of information living on the boundary [82–84]. However, it is important to emphasize the difference between gravity and quantum field theories in the non-gravitational limit and the Hilbert space constructed for them. This way, gravity holds a special difference.

We start by picking an Euclidean vacuum and states can be constructed on the vacuum through fluctuations. In constructing these normalizable states, we have to add smearing functions in QFTs, since without them they would move out of Hilbert spaces. Just like the de Sitter case in sec. 4.1.1, the states would not be invariant under isometries of de Sitter in this local QFT Hilbert space. Since there is no 'Gauss Law' here, we do not have states other than Euclidean vacuum that are invariant under de Sitter isometries. Usually, there is a divergence in the norm as discussed in sec. 4.1.1 which can be eliminated by dividing the norm by the volume of the isometry group. However, such an option is not available for QFT Hilbert spaces. In addition, for QFT, we always can prepare split states [85].

In flat space, the information available at boundary $\mathcal{I}^+$ is also available to past the boundary $\mathcal{I}^+_-$ [84]. Similarly, the information available at the boundary of AdS, is also available in a small timelike band on the boundary. The argument in dS is a bit different. Any region in dS is compact, so it surrounds its complement and the complement surrounds the region. So the information in a region R would also be accessible in the region $\bar{R}$. We do not go into the details on these but holography of information has major implications in understanding information available at asymptotic. de Sitter also has such discussion [9].

4.2.3 Other Holographic Aspects of de Sitter

Finally, we will discuss some recent developments in the subject of de Sitter holography and results, such as achieving a holographic description in the perspective of $T\overline{T}$-deformations to describe Cauchy slice holography [47] and the notion of bulk reconstruction very briefly. In later revisions, we will discuss a recent paper by Cotler and Strominger discussing cosmic ER=EPR [86] and on other important works, such as dS/dS.

Cauchy Slice Holography: We will begin this discussion by reviewing a recent proposal by Regado et al [47].[26] The central approach towards this proposal is that one can define an *irrelavant* deformation given by a $T\overline{T}$ operator to deform the boundary CFT onto a bulk Cauchy slice. To understand this, denote the boundary CFT partition function as $Z[g, \Phi]$ and the deformed partition function as $Z_{T^2}[g, \Phi]$. Then, letting the boundary dual live on $\mathbb{R} \times \partial\Sigma$ and the boundary of a Cauchy slice Σ as $\partial\Sigma$, the general statement of Cauchy slice holography is that the dual CFT for which correlators are identified by the partition function $Z[g, \Phi]$ is T^2-deformed so that the partition function describing WDW states $\Psi[g, \Phi]$ on Σ lives on $\partial\Sigma$. Interpreting states on the CFT side as $|\psi\rangle$ and those on the bulk side by the previously used $|\Psi\rangle$, one has to show that the Hilbert space of the CFT $\mathcal{H}_{\text{CFT}}$ and the Hilbert space of quantum gravity $\mathcal{H}_{\text{QG}}$ are isomorphic to each other. In order to see why this is relevant, we will go through some discussions on the relation between $\mathcal{H}_{\text{CFT}}$ and $\mathcal{H}_{\text{QG}}$, in the sense described by a dual space of distributions K^*, where K is the *kinematic* space formed by the collection of spatial $\Psi[g, \Phi]$. While we will *not* go into details regarding the deformations themselves or the counterterms, we will discuss the relevance of such a setup in the context of the near-boundary limit of a holographic theory.

Asymptotic WDW states in AdS/CFT: The motivation towards *why* deformations are relevant can be adopted from Friedel's work [87], which showed that radial solutions to the WDW equation in the asymptotically AdS regime take the following form, upto rescaling of the metric $\bar{g} \equiv \frac{\gamma}{\rho^2}$ [27]:

$$\lim_{\rho \to 0} \Psi[\bar{g}, \bar{\Phi}] \sim e^{iS[\bar{g}, \bar{\Phi}]} \mathcal{Z}_+[\bar{g}, \bar{\Phi}] + e^{-iS[\bar{g}, \bar{\Phi}]} \mathcal{Z}_-[\bar{g}, \bar{\Phi}] \,, \tag{4.56}$$

where similar to the asymptotically dS analysis considered in section 4.1, the functionals $\mathcal{Z}_\pm[\bar{g}, \bar{\Phi}]$ are CFT partition-like functionals. In the case of the term $e^{iS[\bar{g}, \bar{\Phi}]}$, we identify this to be in AdS counterpart to the universal factor in (4.10). Similar to (4.13), the variation of $\mathcal{Z}_\pm$ gives the conformal Ward identity, so that the variation w.r.t a Liouville field ϕ of $\mathcal{Z}_\pm[e^{2\phi}\gamma] \sim \pm i\mathcal{A}\mathcal{Z}_\pm[\gamma]$.

[26] Note that most of the subsequent discussion is in the background of AdS/CFT, which is where the original argument was made. We will make some independent observations in the case of de Sitter holography by comparing Cauchy slice holography with the results obtained for asymptotic quantization.

[27] Here, by $\bar{\Phi}$ we mean the "rescaled" field Φ, similar to the analysis of Φ in terms of the dilution variable O in asymptotically de Sitter analysis.

In the perspective of asymptotic quantization, the deformation parameter is set to be in the limit of $\rho \to 0$, and we expect that the boundary limit of the RG flow is in the IR. In this way, one obtains a motivation towards the $\mu \to \epsilon$ limit of these deformations, which can be described in terms of the counterterms and a path-ordered deformation parameter $O(\lambda)$ and the corresponding partition function of the CFT,

$$Z^{\Sigma}[g,\Phi] = e^{\mathrm{CT}} \left(\mathcal{P} \exp \int_0^{\lambda} \frac{d\lambda}{\lambda} O(\lambda) \right) Z^{\Sigma}_{\mathrm{CFT}}[g,\Phi] \,, \tag{4.57}$$

where $\mathcal{P}$ denotes path-ordering. The idea of this can be encapsulated as follows: the deformation parameter $O(\lambda)$ is path-ordered to make sense of the flow of λ from the limit $\lambda \to 0$ to the deformed limit $\lambda = \mu$ in the theory space. In this sense, the operator $O(\lambda)$ is irrelevant, and as stated previously, the $e^{\mathrm{CT}}[g]$ are counterterms. This is akin to the term $e^{iS[\bar{g}]}$ in Friedel's work; in the large-det$g$ limit, this precisely reproduces Freidel's result (4.56), that under the limit of this rescaling $\bar{g}$ one gets something that looks like a CFT partition function $\mathcal{Z}[\bar{g}]$. In the above case, however, we have made this a lot more apparent by replacing $\mathcal{Z} \to Z$, to capture the fact that the term $Z^{\Sigma}_{\mathrm{CFT}}$ is *indeed* a partition function located on the Cauchy slice.

Deformations in a holographic theory: One could now provide a slightly clearer picture of how asymptotic quantization and Cauchy slice holography are related when in a "finite" bulk theory[28]. In the sense of a near-boundary limit, the functional $\mathcal{Z}[g,\Phi]$ is referred to as CFT partition-like quantity with some caution, whereas one could rather consider, in the sense of such deformations in Cauchy slice holography, that the asymptotic limit gives us WDW states that arise in the particular limit of boundary deformations, offering a perspective on "finite bulk" physics[29].

Bulk reconstruction: In AdS/CFT, we can make sense of bulk reconstruction by identifying a bulk field $\Phi(\mathbf{Y})$ at a point $\mathbf{Y}$ and identify a reconstruction scheme to find dual CFT operators on the boundary. In the near-boundary sense, one can use the extrapolate dictionary to take bulk insertions near the boundary and use the Hamilton-Kabat-Lyfschytz-Lowe (HKLL) picture to reconstruct the corresponding dual operators. However, in dS/CFT, this becomes a non-trivial question. One could make sense of bulk reconstruction better in the sense of the extrapolate dictionary in dS/CFT, which is different from the *differentiate* dictionary used by Maldacena [88] in that there are not one but *two* conformal weights associated to the bulk operator in duality with the CFT operators at $\mathcal{I}^+$ and $\mathcal{I}^-$. This way, one has two dual CFT operators at $\mathcal{I}^+$ and $\mathcal{I}^-$ that

[28]Note here that there are several issues with trying to consider the Hilbert space result (4.10) as naively as the discussion presents; the interaction Hamiltonian was conveniently dropped out in the large Ω limit, whereas in finite Ω we have to factor in both the interaction Hamiltonian and $1/\Omega$ corrections. Further, we also have to take into account of UV completion, which we have not discussed in this paper. In principle, a dS/CFT analog can be constructed, as stated in the main text, where a finite Ω theory can be constructed by deformations from $\mathcal{I}^+$ whether this could possibly lay further constraints on the uniqueness of $\Psi[g,\Phi]$ or $\mathcal{Z}[g,\Phi]$ is something we cannot comment on as of yet.

[29]That is, finite-volume bulk physics. Infinite bulk physics refers to the asymptotic limits, where the usual form of (4.3) holds.

are related to each other via a Bogoliubov transform. In the sense described in (4.53), we have the construction in terms of a double set of operators at $\mathcal{I}^{\pm}$:

$$\lim_{\eta\to 0}\Phi(\eta)\sim\eta^{\Delta}\mathcal{A}\mathcal{O}^{+}_{\Phi}+\eta^{D-\Delta}\mathcal{B}\mathcal{O}^{-}_{\Phi}\ . \tag{4.58}$$

In the dS/CFT picture, one could attempt a HKLL-type prescription by taking the double light-cone system as usual AdS/CFT and doing something like

$$\Phi(\mathbf{Y})\sim\int K^{\pm}_{\Delta,\,D-\Delta}\mathcal{O}^{\pm}_{\Phi}\ , \tag{4.59}$$

where $K^{\pm}_{\Delta,\,D-\Delta}$ is the smearing function in the spacelike wedge bounded by the double light-cone. In dS/CFT, one can do this while keeping in mind that the CFT operators correspond to $\mathcal{I}^{+}$ or $\mathcal{I}^{-}$. Then, the smearing function $K^{\pm}_{\Delta,\,D-\Delta}$ would have two sets corresponding to the evolution to $\mathbf{Y}$ from $\mathcal{I}^{+}$, and another from $\mathcal{I}^{-}$. Such a prescription was found in [89].

In a way, this also motivates the emergence of the bulk de Sitter from the CFT copies, as it was shown by Cotler and Strominger in their recent paper [86] using quasinormal modes and quantization. The result here is that by doubling the CFT copies at $\mathcal{I}^{+}$ or $\mathcal{I}^{-}$, one can make sense of the bulk from entanglement between two CFT copies. However, we will defer a discussion of cosmic ER=EPR to a later revision.

5 Discussions

By reaching this far, we may declare de Sitter (and dS quantum gravity) a far-enriched problem that would require complexity and a unique solution. Another rather pessimistic possibility is that de Sitter is a notorious question that can only have speculations, at least about its asymptotic solutions. We buy the former as of now. We did not discuss many abstract mathematical foundations and speculation about de Sitter in this review, but one can argue that de Sitter is interesting as an abstract mathematical problem[30] and at the same time it is also a very well observational physics about our universe.

In these notes, we started with discussing the geometric preliminaries of de Sitter spacetime, where we discussed the conformal completion of asymptotic de Sitter spacetimes. The boundary data is given by $(g^{(0)},g^{(n)})$. There are many more interesting questions about de Sitter boundaries that we did not discuss, see for instance, [90].

Next, we studied the entanglement entropy in de Sitter. While AdS/CFT was very convenient for us to make sense of holographic entanglement entropy due to a unitary dual CFT and a timelike boundary, dS/CFT has a very striking issue in that the CFT may not be unitary and instead is characterized by a transition matrix instead of a

[30] The understanding of von Neumann algebra for a static patch of de Sitter is one of analysis coming from mathematics besides geometry.

density matrix, due to which we have pseudo holographic entanglement entropy, which is a complex-valued holographic entanglement entropy. We discussed how this is found by double Wick rotations from AdS/CFT, and how the real part of the pseudo entropy is half of de Sitter entropy. We then discussed the bit threads approach to entanglement entropy in the static patch holography perspective as provided by Susskind and Shaghoulian, which has two descriptions characterized by the fashion in which the bit threads are sourced – namely, the monolayer and the bilayer proposals. We then discussed a semiclassical take on these proposals. An extremal surfaces discussion will be provided in later revisions to this review.

In sec. 4, we discussed two different ideas of quantum gravity in de Sitter. The one-half is about finding WDW solutions in the asymptotically de Sitter solutions which is found to be a universal factor multiplied with a functional, which is very close to partition function in its appearance [8,9]. The functional satisfies usual Diffeomorphism invariance and obeys some Weyl transformations. This was done for the volume of Cauchy slice (the topology was taken as a sphere S^D) tending to infinity. We studied different anomaly equations which also resembled, up to a change, conformal anomalies of AdS/CFT. Then we defined a norm which is divergent for a QFT state, but by using gravity we renormalized the norm.

The other-half of sec. 4 was about finding the von Neumann algebra for states in de Sitter for a worldline in static patch [10]. We have a maximum entropy state ($S = 0$) in de Sitter which is the Bunch-Davies vacuum with thermodynamic properties as discussed. Using a crossed-product algebra ($\mathcal{A} \otimes B(L^2(\mathbb{R}))$) and by bounding the observer's energy $H_{\text{obs}} \geq 0$, the states are of type II_1. If we do not bound the energy, we get the algebra type II_∞, which is the suitable algebra of observables of a black hole horizon. It is interesting to observe that de Sitter's maximum entropy is not exactly zero, so the entropy defined in type II_1 is up to an additive constant.

In these notes, we significantly discussed the idea of states in de Sitter. From here, we hope to find clearer results; for instance, do WDW solutions at finite time slices? Or what is the actual nature of de Sitter holography? There have been many advancements in this field recently, but we are far from a stable answer. A good question among others is to ask, which is related to our discussion of static patch de Sitter, if we can take an observer and a time from t_1 to t_2 on the word line γ, what would be the algebra then?

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A Types of von Neumann Algebra

There are three types of von Neumann algebras, determined by (a) the nature of projections and (b) the nature of the trace function. In the trace classification, one sees that on the basis of trace, there are three vN algebras:

1. **Type $\mathbf{I}_n$:** Defined, finite-dimensional Hilbert space.
2. **Type $\mathbf{I}_\infty$:** Undefined, infinite-dimensional Hilbert space.
3. **Type $\mathbf{II}_1$ and $\mathbf{II}_\infty$:** Defined (for the latter, there exists a "trace class" for which the trace exists).
4. **Type III:** Undefined.

Type I algebras are simply the algebra of quantum mechanics; on the basis of projection, a factor is type I if it has a nonzero minimal projection. Type II_1 factors contain nonzero finite projections, but no nonzero minimal projections. There exists a nonzero trace function $\mathrm{tr} : \mathcal{A} \to \mathbb{C}$ as detailed previously. Type II_∞ factors can be expressed as the tensor product of the form $\mathrm{II}_1 \otimes \mathrm{I}_\infty$. Type III factors contain no minimal projectors and no trace, and is usually the general setting of QFT.

B Gelfand-Naimark-Segal Construction

In this appendix, we will look at Gelfand-Naimark-Segal (GNS) construction [91, 92]. For a complete exposition see [55].

The algebra $\mathcal{A}$ we have defined is a linear vector space in itself over the field $\mathbb{C}$. A state is a complex-valued linear function

$$\rho\colon \mathcal{A} \to \mathbb{C}. \tag{B.1}$$

In other words, for $\mathrm{a} \in \mathcal{A}$, we define an inner product $\langle \mathrm{a} \rangle_\rho$. Assuming that a is positive, these inner products are semi-definite. A positive element a in $\mathcal{A}$ corresponds to having $Spectrum\mathrm{a} \subset \mathbb{R}^+ \cup \{0\}$, which are the non-negative reals. We take all the elements $X \in \mathcal{A}$ with $\rho(X^*X) = 0$ as a linear subspace $\mathcal{N}\mathcal{A}$ of null vectors

$$\rho(X^*X) = 0, \quad X \in \mathcal{N} \tag{B.2}$$

which is a left ideal. This can be named as Gelfand's ideal of states. Therefore, for an element $\mathrm{a} \in \mathcal{A}$

$$\mathrm{a} \in \mathcal{A}, \quad X \in \mathcal{N} \implies \mathrm{a}X \in \mathcal{N} \tag{B.3}$$

which is equivalent to saying

$$\rho(X^*\mathrm{a}) = 0. \tag{B.4}$$

Now the quotient space $\mathcal{A}/\mathcal{N}$ is a pre-Hilbert space and a linear space with a positive definite inner product. An element a in $\mathcal{A}$ is written in $\mathcal{A}/\mathcal{N}$ as the equivalence class [a] modulo $\mathcal{N}$. Now the function eq. (B.1) becomes

$$\mathcal{A}/\mathcal{N} \times \mathcal{A}/\mathcal{N} \to \mathbb{C} \tag{B.5}$$

with a well-defined inner product $\langle\rangle_\rho$ to be positive-definite with no non-trivial null vectors. The Hilbert space $\mathcal{H}_\rho$ is defined as a completion of $\mathcal{A}/\mathcal{N}$ in the norm topology given by the norm $||\cdot||_\rho$ induced by the inner product.

There is, by definition of Gelfand ideal, a natural action by linear and bounded operator Π_ρ of a Banach algebra $\mathcal{A}$ on $\mathcal{A}/\mathcal{N}$ given by

$$\Pi_\rho(\mathrm{a})[\mathrm{b}] = [ab], \quad \forall \mathrm{a} \in \mathcal{A}, [\mathrm{b}] \in \mathcal{A}/\mathcal{N}. \tag{B.6}$$

Thus we can associate to every state $\rho \to \mathcal{A}$ a representation Π_ρ in $\mathcal{H}_\rho$. This is called GNS construction for algebra $\mathcal{A}$.

Note that while constructing GNS Hilbert space $\mathcal{H}_\rho$, we did not have to worry about spacetimes and observers. However, it is not clear clear what it means.

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