September, 2021

Thoughts on Field theory and Beyond

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Abstract

In the following paper, we discuss the idea of field theory and QFTs. We point out some important things about string theory, including ghosts, conformal symmetry, and string theory itself. Random matrices are discussed as well. We have tried to make it as subtle and simple as possible.

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Contents

1 Introduction and Discussion

The idea of field theories is a holistic approach that has its roots in many physics, to name two; high energy physics [1], and condensed matter physics [2]. However, the latter one uses a slightly different field theory strategy compared to high energy physics, which we will discuss somewhere below.[1] One striking and most important similarity between Quantum Field Theory (QFT) and Statistical Field Theory (SFT) is Renormalization Group (see Wilson's [3]). Furthermore, when we have a $\mathcal{H}$, a Hilbert space, we can try to generalize the properties of it using ordinary quantum mechanics briefly. But to learn about the fields, which is a broader perspective to understand the physics of a particular regime, we need to consider the theory which will allude us to talk about the generalization of free and interacting models of that regime. So, that is why we speak of field theory in general. If we try to adopt Weinberg view of QFTs [4], the definition of QFT can be explained using three basic principles, which are also the necessity of any theory to be called QFT. These are an S-matrix, Lorentz invariance, and cluster decomposition.

Generally, a field theory is not quantized until it is made because roughly any field theory is an infant classical theory.[2] Some classical field theories, for instance, Einstein's Gravity, face inconsistencies and constraints when we try to quantize them using regular quantization (however, there are some ways to quantize gravity, for instance, in string theory [5–8] which is a question and part of a much larger query of quantum gravity).[3] Some field theories are easily quantized; an elementary example is classical scalar field theory, which we will denote as φ. This scalar field theory for some time will entertain us. In what follows, φ is a free field, excluded from any kind of external perturbation and interactions, except for one example it will be otherwise.

Any field theory, take Newton's field, we have some equations of motion describing the field. For a classical version of field theory we can write those equations using "*Hamilton Mechanics*". And for a quantum version, we use "*Lagrangian Mechanics*" (or action). Both are extremely powerful tools. Note that Hamiltonian dynamics are also used in quantum field theory. So, how can we write a linear classical theory of φ with some mass "m"? For such, we write the Lagrangian (or action)

$$\mathcal{L} = -\frac{1}{2}\left(-\eta^{\mu\nu}\partial_\mu\partial_\nu\varphi + m^2\varphi^2\right) \tag{1.1}$$

[1] A third, but mathematical, one is algebraic field theory (or axiomatic field theory) which deals with operator algebras of field theory. We will see few example of them in this paper.

[2] This statement is party wrong as there are many field theories which has quantum symmetries as well, even if their action is classical.

[3] We do not wish to particularly talk about the issue of quantization of gravity in this paper.

where we work in $(+ - --)$ signature[4], $\partial_\mu = \frac{\partial}{\partial \mu}$ and $\partial^2 = \partial_\mu \partial^\mu$. It is explicitly seen that in Lagrangian (1.1) that it is a relativistic theory. We can now write the classical Klein-Gordon equation[5] related to φ using "Euler-Lagrange Equation"

$$\eta^{\mu\nu}\partial_\mu\partial_\nu + m^2\varphi = 0. \tag{1.2}$$

The Euler-Lagrange equation is a differential equation (much like the Schrödinger equation but more powerful in the sense of field theory machinery), we write them naively as

$$\frac{\partial \mathcal{L}}{\partial f_i} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial \mathcal{L}}{\partial f_i'}\right) = 0, \tag{1.3}$$

where f is some function[6]. We solve the differential equation (1.3), and then we get an equation which, sometimes, is sufficient to know much about the field. Eq. (1.2) is a classical solution, and we want to know the quantum properties of the φ. There where the "quantization" comes. Quantizing fields go back to electrodynamics, see [9]. A very successful model of quantization of classical field, at that time, was only "Quantum Electrodynamics" (QED). Quantization can be thought of in mainly two ways: Canonical Quantization and Path Integral Quantization. Both are important ways of quantization. For instance, finding Instantons in quantum theories can be done using path integral approaches, and most of the fields can be easily quantized using canonical variable and oscillators setup. String models, a very proclaimed theory for quantum gravity, can also be quantized in both ways[7].

The outline of the paper is as follows; in the ongoing **section 1**, we are discussing and mainly introducing various quantum field theories methods, which include Lagrangian, quantization, topological quantum field theories, statistical fields theories methods, Feynman methods, and green functions. In **section 2**, we will discuss the symmetries of nature and their basic properties, which include current, large N, generators, and so on. We will briefly comment on Yang-Mills and duality in the presence of Large N. In the subsection 2.1, we will discuss the gauge theory and what we mean by gauging a quantum field theory and what ghosts are. We will also briefly discuss the two kinds of ghosts, namely covariant- and non-covariant - ghosts. In subsection 2.2, we will discuss the invariance and group theory.

In **section 3**, we will talk about string theory in nutshell. Mainly we will discuss its interpretations and its usefulness using light coordinates. In the succeeding subsections of the same section, we will try to discuss Virasoro algebra and conformal algebra (subsection 3.1), Ghosts and BRST quantization (subsection 3.2), and few other basics (subsection 3.3). In **section 4**, we will outline few basic properties and definitions in Random Matrix Theory. Finally, in the in **Appendix A**, we will discuss few geometries of Anti-de Sitter space.

To establish a canonical quantization of φ, we have some procedure. (We will not mention all of them here.) An important one is to write the commutation relations for x and p for the field.

[4] Also $c = \hbar = 1$, which are natural units in QFT.

[5] This is a classical solution (1.2), however, we can say it is a Klein-Gordon equation. But not a *wave* equation. For that, we need to quantize φ and then use Euler-Lagrange equation.

[6] In most of the quantum cases, the function is of canonical variables.

[7] Path integral approach will be replaced by Polyakov integral, a good discussion can be found in [10].

Then find the oscillators of the field, after establishing a *vacuum* $|\phi; 0\rangle$. These oscillators will excite the vacuum to first excited state or it can oscillate the excited state to lower state (or vacuum for some examples). Conventionally, they are a_n and $a_n^\dagger$, where n is some integer. They are also called annihilation operator and creation operator respectively. Then we do some ordering, and our $\mathcal{L}$ is ready. For φ, when quantized, the action is given in terms of[8]

$$\mathcal{L} = -\frac{1}{2}\left(-\partial^2\varphi^2 + (m\varphi)^2\right), \tag{1.4}$$

and given the Euler-Lagrange (1.3), we can write;

$$\partial^2\varphi + m^2\varphi = 0, \tag{1.5}$$

which is a simple linear equation and shows a Schrödinger equation for wave function φ. For a set of scalar fields, we can write the Lagrangian for φ as

$$\mathcal{L} = -\frac{1}{2}\sum_i^N\left(\partial^2\varphi_i^2 + m^2\varphi_i^2\right) - \frac{1}{8}\lambda\left(\sum_i^N\varphi_i\right)^2, \tag{1.6}$$

which of course is interacting field and in (1.6) λ is coupling.

If we dig more to group theory, we find that under the large $N \to \infty$ of the group, interesting things happen, general discussions by 't Hooft can be found at [13, 14]. (Interestingly, $SU(N)$, which represent QCD at $N = 3$, when formulated at $N \to \infty$ becomes a free string theory with coupling $1/N$ [15]. Historically, the theory of string theory originates from discussion of hadrons and mesons themselves [8].)

That is a preliminary definition of *fields* and quantization of which main motive is to describe the states-oscillations.

A rather comprehensive approach would be topologically [16–18]. The most exciting application of them is "Chern-Simons" Theory - a topologically invariant theory (or equivalently metric invariant) - first taken to general QFTs as a topological solution by Donaldson [19, 20], and Witten [21] using the works on knot polynomials. In a topological quantum field theory (TQFT), we rather want to know the geometries of the fields, and we analyze them in terms of rigorous topological mathematics. Generally, TQFT is an extension of topological operators of higher forms. *Genus* is an important property of TQFT when we study them. Genus, denoted by g, is a topological form of describing the, roughly speaking, number of holes in geometry[9]. One can describe the expansion of some fields using genus expansion (where the genus is for closed surfaces). Writing these genus expansions is valid for any field theory. However, the most interesting case is of 't Hooft limit ($N \to \infty$), as an equation is formed with Euler characteristics, which resembles the equation for genus for closed-oriented surfaces at the instance.

[8] This scalar field theory have scale invariance and conformal invariance [11]. A very typical type of scalar invariance in QFT is studied in [12]. One can also speak that when beta functions, $B(g) = \partial g/\partial log(\mu)$, the field theory is scale invariant.

[9] Sphere has $g = 0$, whereas torus has $g = 1$, see figure 1.1.

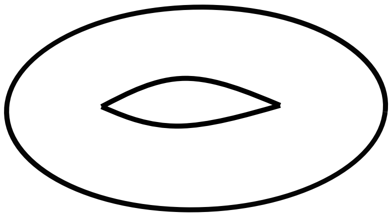

Figure 1.1: A (2D) torus with $g = 1$.

A Short Review of Topological Quantum Field Theory: While a lot of the ideas in TQFT relate to the ordinary quantum field theory. Instead, we will try to define the usual definition of TQFT independently, following Segal's definition [17, 22].

There are many other things in TQFT, a somewhat different approach from general QFT [21]. For example, interpretation of differential forms, Jones polynomial, and Langlands Program [23]. Or the interpretation of Chern-Simons theories in Quantum Hall Effect [24]. (Just as we have AdS/CFT correspondence, we also have a correspondence between Chern-Simons theory and WZW [21, 25] on the boundary, which is a correspondence between topological field theory and non-topological field theory.) One also feels a proper geometrical (and categorical) *spacetime* in such theories.

(Another important topic would be closed curve solution to N holes genus in Moduli spaces, extensively studied by Eskin and Mirzakhani [26]. And studying the shapes of that genus is partly defined in TQFTs.)

A somewhat distinct approach would be *statistical field theory*. But we only study SFT to understand the statistical significance of such fields. And there are many similarities between QFTs and SFTs, sometimes it can be hard to distinguish between the two. In SFT, we have partition function for some Hamiltonian H

$$Z = \sum_n e^{-\beta E_n}, \tag{1.7}$$

or equivalently as

$$Z = tr(e^{-\beta \hat{H}}). \tag{1.8}$$

In QFT, one also has such a partition function with a slightly different definition. In SFT, all you want to know is the partition functions, and then you can compute many things with it. In QFT, partition functions are generally used to understand the *Green functions*, and other Feynman approaches. Moreover, Feynman's methods are the essence of any field theory. This includes; path integrals [27], Feynman diagrams, Amplitudes, and many more. A path integral ($\mathcal{D}[x]$, where X are paths) is basically an integration over all possible paths, same as partition function (1.7) is a summation over possible energies. And in QFT, a partition function is simply a generating functional (with some source J coupled to the field) that generates all correlation functions (We should also mention here that Lagrangian is not only the conventional approach to study QFTs, one can also study it by listing all the correlations functions. And there are still more methods.).

So, we reach

$$Z[J] = \int \mathcal{D}[x]\varphi e^{\left(iS[\varphi]+\int d^4x J(x)\varphi(x)\right)} \tag{1.9}$$

where S is action of the field. When $J = 0$, that means no source energy coupled to field, then $Z[J = 0]$

$$Z[J = 0] = \int \mathcal{D}[x]\varphi e^{iS[\varphi]}, \tag{1.10}$$

where (1.9) is of course showing the coupling of J with the field in dimensions $D = 4$.

Green functions are propagators of the theory. Let us say that φ is locally and internally interacting. And when φ goes to y from x, there is some propagator making it possible, we will say it is $G_{x,y}$. Before we make remarks on that, we need to discuss the Feynman calculus of our field; the field in our case is φ. Actually, we should not be writing the calculus first and then learn the field, and it should have been reversed. However, we know the action (1.6), and we can know the basic propagators of scalar field theory without deriving them here. There are some Feynman rules for every field. From our quantization process, we can derive φ in terms of oscillating modes

$$\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{2E_n} \left(a_{\boldsymbol{n}} e^{-ipx} + a^{\dagger}_{\boldsymbol{n}} e^{ipx} \right), \tag{1.11}$$

where we integrate over momentum. It is important to note that we have localized the φ as a function of x.

We can draw the Feynman diagrams for (1.11) and S-matrix for interactions. It is evident by name that the excitations of φ are scalar particles. In an interacting theory of φ, J is only dynamical and do not self-interact. Let suppose α_φ and β_φ are two scalar particles of free φ. We want to know things about the scattering $\alpha_\varphi\beta_\varphi \to \alpha_\varphi\beta_\varphi$. Feynman diagram of this scattering on tree level is figure 1.2. Where two particles come and interact and then scatter away. Every line in the figure is the history of that particle. (In the whole process, γ_φ is some mediator which we do not show here. IT is not a photon, and the wiggling line is just for convenience. The photon would be present in interactions involving QED.) These mediators play the role of bosons. To write the "Feynman Rules", which is distinct for different fields and configurations, we report the amplitude of the scattering. In order to note that, we need to know the amplitudes for vertices, propagators, and particles themselves. In higher-order, we will delightfully need loops amplitude also. We suspect that real nature has all the higher-order loops for interactions. For brevity, we generally do not include higher-order terms. However, one can start with real nature and integrate the higher orders (or higher mass orders) using Renormalization.

Once we are done with writing Feynman rules, we know the theory much well. Nevertheless, still, we want to know the S-matrix associated with figure 1.2. We write it as

$$S_{mat} = \langle \mathcal{A}_{Out} | \mathcal{A}_{In} \rangle \,, \tag{1.12}$$

where $\mathcal{A}$ is the amplitude. In the discussion of the S-matrix, we get to know the amplitude change in the process. Moreover, once we know S-matrix (or matrices), we get new pathways to understand more about those interactions.

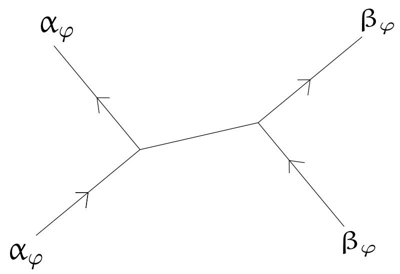

Figure 1.2: A Feynman diagram among many, a tree level $2 \to 2$, for $\alpha_\varphi \beta_\varphi \to \alpha_\varphi \beta_\varphi$. It is important to note that this is not the *only* way to draw diagrams, we can map this diagram (conformally) to some disk or sphere and work on those diagrams also.

Of course, φ is not a natural field that excites matter, QCD is one of this kind, but φ can be a good starting toy model (One example of the scalar particle is Higgs Boson.) in the process of making the Standard Model (SM) (see, for instance, [28]), which is the most influential field theory that comes from fields and plays with things related charge, color, global symmetries, discrete symmetries and so on.

2 Symmetries and Groups

Symmetries of the nature, though they are not necessary to be "natural", are a delightful feature of QFTs (or physics in general). What we intend to study in symmetries is reflected in the study of the field's internal working and much more. We will mostly talk about QFTs symmetries in this section. The most naive way to understand symmetry lies in the application of conservation, as two words are conjugation to each other. In other words - conserved charges are generators of symmetry. For instance, Noether's Current is a conserved quantity and, roughly speaking, it corresponds to a continuous symmetry.

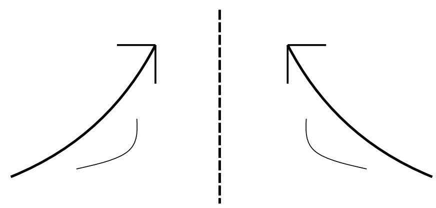

Figure 2.1: Showing chiral symmetry here. The curve on both sides, let us suppose it is a particle, should be the same. A mirror will not change the particle to a new kind; it will still be the same, i.e., laws should be conserved.

Generally, every QFT has some generators. These generators, in terms of group theory, can generate other components of that group. Group theory is another interesting mathematics. These groups are supposed to be a fundamental (or even composite) description of the field. If we speak straightforwardly, groups are another essence of fields. Consider a system with some matrix that has determinant 1, which - suppose - is a unitary system. For such system, we can associate a group called **SU(N)**. Every group has its algebra and representations. For instance, SU(N) has $N^2 - 1$ number of elements. QCD, which is a theory of $SU(3)$, has eight elements, which are actually the colors of the theory.

If we talk about large N, suppose $N \to \infty$ - of some part we discussed in section 1 - then good things happen and system behaves a little - delightfully - peculiar. As we mentioned that $SU(N)$ for large N behaves like a free string theory, a complete review can be found in [15, 29]. Moreover, we get the perturbation expansion in terms of coupling $1/N$ [30]. This is not only interesting in $SU(N)$ but groups like $U(N), SO(N)$ also have interesting points to make when $N \to \infty$. This large N is an important contribution for high energy physics (and condensed matter physics), though, despite the large degrees of freedom which it brings in. Recently, we have seen a lot of 't Hooft limit applications.

In modern-day physics, there are many dualities of concern. String/Gauge duality is also interesting. However, major developments have already been made in past years. For an interesting read of String/Gauge Duality see [15,31]. This duality is only of importance when N is large enough, identified as *'t Hooft limit* [13], as only Large Ngauge theories are said to be String theories. The explanation of the cause is, of course, in detail.

Now, we will briefly comment on "Linear Algebra", which is another well mathematical algebraic topic and has preliminary roots in quantum mechanics as well. This is a well-studied subject and has its implication in various works of QFTs. There are many things in Linear Algebra, two of the important are *eigenvalue* and *diagonalization*. In studies of QFTs (or even basic Quantum Mechanics), this simple algebra is well defined and used for every bread and butter making. To mention one basic concept called commutation, a commutator between two generators of the group is the subject of that group and forms an algebra for the same. In general, a commutator is defined as

$$[A, B] = AB - BA. \tag{2.1}$$

Elements of the **abelian group** are commutative, i.e a and b are abelian group elements then

$$ab - ba = 0 \tag{2.2}$$

It is otherwise for non-abelian group elements, i.e., they do not commute. The latter (non-abelian group) is a study theme for Yang-Mills theory [32]. Yang-Mills (YM) theory has various implications in particle physics (both theory and phenomenology) of **gauge**[10] **groups** [33] - a condensed matter review is [34]. A supersymmetric extension for YM is Super Yang-Mills (SYM)[11]. Our QCD, an asymptotically free theory, is a Yang-Mills theory. Different supercharges ($\mathcal{N}$) produce different SYM, the most interesting of them is $\mathcal{N} = 4$ SYM, which is a perfect model to study, as we will see, the correspondence between a string theory and gauge theory. The essence of YM is gauge invariance, which we discuss next.

[10]Discussed in next section.

[11]Some reviews can be found in [35] with few applications.

2.1 Background on Gauge Theory

We will now discuss, briefly, the basics of the gauge theory. However, a detailed discussion can be found in [36]. We will mostly follow notations of [37]. Gauge theory is a theory of gauge fields and their related transformation. A "gauge" is just a simple choice. Furthermore, a gauge field is a vector field with continuous parameters defined by the gauge group; for example, a YM field with $SU(N)$ and a gauge field $\mathcal{A}_\mu$ is a gauge theory. When we write some gauge for a theory, we are building the theory in those gauge properties. Let us suppose we have a field Φ (of N copies of φ) which transforms according to a Lie group - Lie groups are dependent on a set of continuous parameters - and suppose the group is a compact one. That group, G, will have some generators. If the transformation from G is carried out *locally* i.e., it depends on space-time variables, and then we have a local gauge group. If it does not depend, then we have a global gauge group. For the local gauge group, we define the gauge field and fix the overall theory. So if for the field Φ and group G we have k^α as infinitesimal transformation function and g is an element of group G with generators a, then transformation is given by

$$g = g_0 + k^\alpha a^\alpha, \tag{2.3}$$

where k^α is locally dependent. As we have said above that if k^α is locally dependent, then we need to introduce gauge field $\mathcal{A}_\mu$ which transforms like

$$\mathcal{A}_\mu(x) \to {}^g\mathcal{A}_\mu(x) = g(x)\mathcal{A}_\mu g^{-1}(x) + [\partial_\mu g(x)]\, g^{-1}(x), \tag{2.4}$$

and that is how a gauge theory is defined. We can now fix the action related to Φ with inserting this gauge field into it. Eventually, we will get a gauge theory with gauge invariance in it. As we mentioned, if a theory is gauge invariant, then it has gauge freedoms which help to study the model effectively (and write helpful caveats).

There are two classes of gauge theory (at least in terms of consistencies and their "ghosts" or "ghosts free" nature) - covariant and non-covariant gauge theory. Both are famous and infamous for their own reasons, exploited in [37]. Actually, covariant gauges have a history with ghosts. Technically, a ghost particle (or state) is an unphysical particle (or state) which are sometimes necessary to eliminate the ill-amplitude that arises from closed loops. However, there are bad ghosts also, which contributes to the negative energy spectrum[12]. Good ghosts include the famous *Fadeev-Popov ghosts* and bad ghosts would be *Landau ghosts*. A goldstone state can also be considered a ghost.

The table 1 and 2 in [37] principally shows the covariant and non-covariant gauges and follows the discussion, in same [37], about those gauges choices with faithful explanations, that we will not do about covariant and non-covariant gauges in this paper, except that light-cone gauge in a later section. However, in what follows, the invariant statement is that covariant gauges have ghosts while non-covariant have not.

[12]Not the hole-theory way. They contribute to negative kinetic energies. Also, note the distinction between a tachyon and a ghost; the former has mass-squared as negative. Tachyons generally appear in the ground state of bosonic string theory and are unstable.

2.2 Invariance and Groups

This section will touch on the invariance principle and a little bit of group theory in this section. In relativistic physics, Lorentz invariance prevails the whole discussion of symmetry, at least in GR and most of the QFT. In a basic sense, Lorentz invariance preserves physics through space-time. We understand Lorentz invariance as a necessity for reality, as it appears. In general relativity, Lorentz transformation is a change of frames (or perhaps indices). If the transformation does not alter the physics of the previous frame to the new frame, we call the transformation "invariant".

As we said that every invariance is associated with a group algebra. Lorentz group (in $D = 4$), a group for Lorentz transformations, is $SO(1,3)$. Furthermore, the covering group for same is given by a special linear group[13] $SL(2, C)$. We will mention the generators of the Lorentz group as $J_{\mu\nu}0$. Lorentz symmetry is global symmetry. Roughly speaking, if this group acts and leaves action of the group invariant, then it poses no inconsistencies in Lorentz invariance for that particular field theory. A Lorentz invariant theory is no threat to absurdity. For instance, every SM QFTs are Lorentz invariant (and Superstring theory (a theory for supersymmetric strings) is also a critical theory at $D = 10$ as it is Lorentz invariant in only those critical dimensions). We frequently see the high importance of Lorentz invariance (or extended Poincare invariance) in physics. There are more groups of reliance, such as Poincare group and Conformal group [38–40]. The latter is a great deal and is currently involved in breakthrough achievements and developments.

Let us see some of the properties of the transformation of a manifold $\mathcal{M}$. Suppose $\mathcal{M}$ is a regular manifold. For a point ϵ on $\mathcal{M}$, translation invariance holds that for transformation

$$\epsilon \to \epsilon + x, \tag{2.5}$$

that both points "ϵ and $\epsilon + x$" have the same laws of $\mathcal{M}$. If we include Lorentz boosts and maps together with these translations, we call it Poincare invariance. Furthermore, Lorentz invariance is a symmetry holding that the change of frames (and translations too in Poincare invariance) do not change the defined laws. In the covariant description of relativity, these Lorentz transformations were first physically defined by Einstein. For a general Lorentz transformation, we write

$$x \to x' = \Lambda x, \tag{2.6}$$

where Λ is a matrix of all Lorentz transformations. Moreover, if this transformation preserves the physics, then (2.6) is Lorentz invariant (or Poincare invariant if translation included). As usual, these invariances are connected with symmetries. And these all (translation, Lorentz, and Poincare) have their own groups and representations.

In particle physics, we deal with many symmetries; some are broken, and some remain unbroken. The Famous of them is **CPT** symmetry, and they stand for *charge conjugation, parity, and time reversal* (all three are discrete symmetries). If we do CPT transformation on a particle e (not necessarily electron), then e now becomes opposite charged, with opposite parity and with the opposite timeline in the Feynman diagram. We can, of course, take only of the three at one time as well. Among CPT, when they are individually performing, two have been observed to be broken

[13]SL(2,Z) is a different group which is for modular transformation. Modular transformations are important in string theory from the world-sheet point of view.

as symmetry; for instance, parity invariance has been observed as broken in weak interactions. There is also the special chiral symmetry in SM which is one of the key features in Electro-Weak theories (a $SU(2) \times U(1)$) [41, 42]. There are not only these cases of broken symmetry. Studies around the baryon symmetry and asymmetry have been done, and it asserts why there is an unequal distribution of baryons and anti-baryons in the universe. That is why particle physics phenomenology keeps an eye on every symmetry, either broken or not. In the above discussion, we already realized that symmetries and group theory are linked strongly.

As we are inter-changeably discussing invariance and group, we can briefly talk about some groups. It is well note fact that gauge theories describe every SM theory. One such example is $U(1)$, which is the group for Electromagnetic theory. This is a unitary theory with $N = 1$. Furthermore, $SO(2)$ describes weak theories with $N = 2$ in three dimensions, and $SU(3)$, as we have described, represents the strong theories of QCD with 8 elements with $N = 3$. Nevertheless, the discussion is not limited to SM theories, and we may want to build some good-old theories with $Sp(p, q)$, $SL(2, Z)$, and so on. Indeed, we have made-models, and we can instantly think of string theory, particularly those various types of world-sheet group theory. It has also been made possible only because of the mathematical intuitions, which are plenty in quantum gravity, such as string theory.

3 String-y Theory

Now we turn to a proclaimed quantized theory in which gravity is inevitable. We said in section 1, gravity can not be quantized using "usual" quantization processes. That is a failure of quantum field theory. However, in large part, it has been quantized using string theory (for a general introduction, we refer to [43, 44]). The string theory's consistencies with Lorentz algebra are only available in critical dimensions and a number called "a". Nevertheless, it is worth mentioning here that non-critical strings theories are also of good importance. In what follows, we will only focus on critical strings theories. For brevity, we mostly follow notations from [43].

Gauges were mentioned in subsection 2.1, of which we said that sometimes non-covariant gauges can help us to eradicate the ghosts. Light-Cone gauge is one of those kinds. String theory is comfortable in light cones. For X^μ, we introduce the light-cone coordinates as

$$X^\mu = \frac{1}{2}(X^0 \pm X^{D-1}), \tag{3.1}$$

where D are spacetime dimensions. Eqn. (3.1) is stating that two coordinates (X^0 and X^{D-1}) have been selected out in a non-covariant way. The remaining coordinates are called transverse coordinates $X_i, i = 1, \ldots, D-1$. For any two vectors V and W, we write inner product as

$$V \cdot W = V^i W^i - V^+ W^- - V^- W^+. \tag{3.2}$$

Action: String theory has its origination from Dual-Models [45] studies on the Regge behavior of hadrons and mesons. What was believed to be a theory of QCD was actually a string's theory. Later, QCD was studied as a YM $SU(3)$ theory [46].

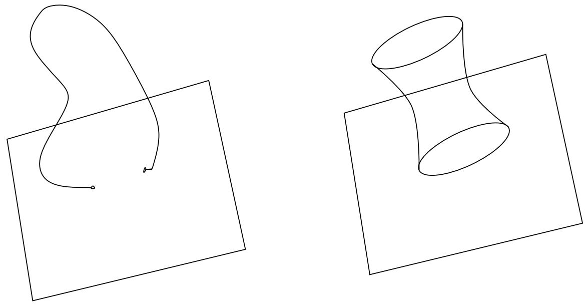

Figure 3.1: (a) Open strings end on D-brane and (b) Closed strings has a source from D-brane.

A model called "Veneziano Model" exploited the idea of duality between the s-channel and t-channel [47], later Veneziano theory became the algebra for open strings (strings with end points). To go from ordinary particle physics to string theory, we replace the known worldline through something strings sweep called *worldsheet*. And strings are one dimensional structures, ending or originating from D-branes[14]. D-branes are also solitons in string theory. To some extent methods of particle physics is applicable in string theory, as so that we can now construct an action for strings in an analogy. An action, known as Nambu-Goto action, is

$$\mathcal{S} = -T \int d^{n+1}\sigma \sqrt{g} \tag{3.3}$$

where σ is some representation and g is metric. But a more developed version of Nambu-Goto action is called "Polyakov action", which is written as

$$\mathcal{S} = -\frac{T}{2} \int d^{n+1}\sigma \sqrt{h} h^{\alpha\beta}(\sigma) g_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu, \tag{3.4}$$

where T is string tension and is written $T = (2\pi\alpha')^{-1}$ where α' is Regge slope parameter. $h_{\alpha\beta}$ is yet another metric, $h^{\alpha\beta}$ is inverse of $h_{\alpha\beta}$ and h is determinant of $h_{\alpha\beta}$. $h_{\alpha\beta}$ is the geometry of $n+1$ manifold and $g_{\mu\nu}$ (an induced metric) is the geometry of D-dimensional space-time. X^μ are maps from world-line, world-sheet to physical space-time. We have a constraint here that $D \geq n+1$. We could have also derived the (3.4) using an arbitrary parameter $e(\tau)$ and then writing the string tension, but our thoughts are linear.

σ is there for reparametrization. (3.4) parts are invariant volume elements. $h_{\alpha\beta}$ is also Weyl invariant, a necessity for eliminating conformal ghosts, which is possible in $n = 1$ as $d^2\sigma$ corresponds to some 2D CFT. Weyl scaling is

$$h_{\alpha\beta} \to \Lambda(\sigma) h_{\alpha\beta}. \tag{3.5}$$

(We should also mention that string theory has two forms of excitations; open and closed strings. In fig 3.1, it has been drawn. They have their own properties (however, lots of overlaps) and algebras.)

[14]Open strings boundary end (Drichlet Boundary Condition is hence satisfied) on D-branes. D-brane is a source for closed strings, see figure 3.1.

3.1 Conformal Group and Virasoro Algebra

Conformal fields theories are another essence of string theory. In fact, we can consider 2D CFT a straightforward model to show our string theory. Virasoro algebra is an algebra for conformal theories developed by Virasoro [48]. There is also intricate algebra like Witt algebra. We should first understand conformal algebra to a necessary extent. A conformal group is a simple group including Poincare and scale-invariance group. The most simple scale-invariant theory, other than the scalar field, is YM in $D = 4$. Scale-invariant theories are - by default - believed to be conformal invariant.

Conformal group is a group of transformations which preserves the metric structure under some arbitrary scaling, it also includes the inverse transformation. This can be written as

$$h_{\alpha\beta} \rightarrow \Omega^2 h_{\alpha\beta}, \tag{3.6}$$

and general scale transformation as

$$x^\mu \rightarrow \lambda x^\mu, \tag{3.7}$$

and special conformal transformation, which is a fractional transformation, is

$$x^\mu \rightarrow \frac{x^\mu + a^\mu x^2}{1 + 2x^\nu a_\nu + a^2 x^2}. \tag{3.8}$$

We can denote the generators for these transformations as follow; P_μ for translations, $J_{\mu\nu}$ for Lorentz transformation, D for scaling transformation, and $K_{\mu\nu}$ for special conformal transformations[15]. They obey the conformal algebra

$$\begin{gathered} [J_{\mu\nu}, P_\rho] = -i\left(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu\right); [J_{\mu\nu}, K_\rho] = -i\left(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu\right) \\ [J_{\mu\nu}, J_{\rho\sigma}] = -i\eta_{\mu\rho}J_{\nu\sigma} \pm \text{ permutations }; [J_{\mu\nu}, D] = 0; \quad [D, K_\mu] = iK_\mu; \\ [D, P_\mu] = -iP_\mu; [P_\mu, K_\nu] = 2iJ_{\mu\nu} - 2i\eta_{\mu\nu}D, \end{gathered} \tag{3.9}$$

and we can write the action of the algebra on a state as [49]

$$[P_\mu, \Phi(x)] = i\partial_\mu \Phi(x), \tag{3.10}$$

$$[J_{\mu\nu}, \Phi(x)] = \left[i(x_\mu\partial_\nu - x_\nu\partial_\mu) + \Sigma_{\mu\nu}\right]\Phi(x), \tag{3.11}$$

$$[D, \Phi(x)] = i(-\Delta + x^\mu\partial_\mu)\Phi(x), \tag{3.12}$$

$$[K_\mu, \Phi(x)] = \left[i\left(x^2\partial_\mu - 2x_\mu x^\nu\partial_\nu + 2x_\mu\Delta\right) - 2x^\nu\Sigma_{\mu\nu}\right]\Phi(x), \tag{3.13}$$

where Δ (scaling dimension) and $\Sigma_{\mu\nu}$ are generators of little groups [49]. We can also mention that Weyl invariance is isomorphic to group $SO(2,d)$, something similar to $SO(1,3)$ Lorentz group (see 2.2). Generators for $SO(2,d)$ will be identical to $J_{\mu\nu}$.

After this basic discussion about conformal algebra, we turn to its most (and famous) important application in string theory, namely "Virasoro Algebra" (see section 2 in [43]). It is an extension of Witt algebra. For L_m generator[16] of Virasoro algebra, we write

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \tag{3.14}$$

[15] We use notations from [15] expect $J_{\mu\nu}$.

[16] They are simply the fourier modes of energy-momentum tensor, that later will produce "Virasoro Constraint".

$$[L_m, L_n]_{P.B} = i(m-n)L_{m+n} \tag{3.15}$$

where c is central charge and $P.B$ means Poisson bracket. While (3.14) is quantum, (3.15) is purely classical. Central charge is an operator with commute with all other operators in the group i.e. $[L_n, c] = 0$. For the unitary representation of the algebra, we write (see [50])

$$L_n^\dagger = L_{-n}. \tag{3.16}$$

One can write L_m in fourier modes (in terms of oscillators that are present in any quantum theory)

$$L_m = \frac{1}{2}\sum_{-\infty}^{\infty} \alpha_{m-n} \cdot \alpha_n, \tag{3.17}$$

(similar for $L_m^\dagger$) with an exception for L_0

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n, \tag{3.18}$$

because when $m = 0$, both oscillators (α_{m-n}, α_n) commute for L_0. Now this L_0 gives a condition to be cared for physical states

$$(L_0 - a)\left|\chi\right\rangle = 0, \tag{3.19}$$

where a is a very critical number for criticality of string theory and χ is a physical state. When $a = 1$ string theory produces smooth states.

An open string with mass squared $M^2 = -p_\mu p^\mu$ can be written with the (3.18)

$$M^2 = -2a + 2\sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n, \tag{3.20}$$

where one can see that $-2a$ is the mass squared of ground state. In (3.20) the value of α_0 is 1/2. Similarly for closed strings[17]

$$M^2 = -8a + 8\sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n = -8a + 8\sum_{n=1}^{\infty} \alpha_{-n}^\dagger \cdot \alpha_n^\dagger. \tag{3.21}$$

The fact that (3.21) is 4 times the (3.20) is another way of saying that $g^2 = \kappa$ where g is coupling constant for open strings and κ for closed strings. One final basic takeaway from Virasoro algebra is that for a physical state χ

$$L_m\left|\chi\right\rangle = 0, \quad m > 0, \tag{3.22}$$

and this is a necessary condition, along with (3.19).

An efficient application of Virasoro algebra is DDF formalism. DDF stands for Di Vecchia, Del Giudice, Fubini [51], who were involved in studying this formalism. DDF formalism includes only physical states, which help us to stand them out from negative states. For a complete description of DDF states, see elsewhere [43].

We suggest the read [52].

[17] Important thing to note that $L_m^\dagger$ are not present in open strings. For closed string, we have $L_0 = L_0^\dagger$.

3.2 Fadeev-Popov Ghosts and BRST

Ghosts (mainly good ones) are sometime necessary in order to fix the gauge which is also known as "gauge fixing". We have to introduce these kinds of ghosts through adding extra fields, which we call ghosts fields. One of them is *Fadeev-Popov ghost*, mainly due to Fadeev and Popov [53]. This ghost is necessary to maintain the path integral formulation of string theory. For a string theory, we write the Euclidean path integral

$$Z = \int D[h(\sigma)]\ D[X(\sigma)]e^{-S[h,X]}, \tag{3.23}$$

where S is action (3.4) and $\int D[h]$ includes integral over three components h_{++}, h_{--}, and h_{+-}. The gauge choice

$$h_{\alpha\beta} = e^{\phi}\eta_{\alpha\beta}, \tag{3.24}$$

then comes the gauge fixing, as described in 2.1. What comes after this is we use the equations for δh_{++} and δh_{--}, we have the famous identity (see [6] for clarity)

$$1 = \int Dg(\sigma)\delta\left(h^g_{++}\right)\delta\left(h^g_{--}\right)\det\left(\delta h^g_{++}/\delta g\right)\det\left(\delta h^g_{--}/\delta g\right) \tag{3.25}$$

and insert it to the path integral, we gain

$$Z = \int Dg(\sigma)\int Dh(\sigma)DX(\sigma)e^{-S[h,X]}\delta\left(h^g_{++}\right)\delta\left(h^g_{--}\right)\det\left(\delta h^g_{++}/\delta g\right)\det\left(\delta h^g_{--}/\delta g\right). \tag{3.26}$$

Because of the reparametrization invariance of $S(h) = S(h^g)$, we can write $h^g = h'$ and h to g. Thus, our gauge fixed path integral is

$$Z = \int Dh'(\sigma)DX(\sigma)e^{-S[h',X]}\delta\left(h'_{++}\right)\delta\left(h'_{--}\right)\det\left(\delta h'_{++}/\delta g\right)\det\left(\delta h'_{--}/\delta g\right). \tag{3.27}$$

Where do we need ghosts? Which are most of the time unphysical. Since (3.27) is gauged, it will contain many irrelevant states in it. We need to distinguish the physical configurations and gauge orbits in them. That is where we introduce ghosts (which naturally arise). In this very case of gauge theory, one has to work with Fadeev-Popov ghosts. It must be noted out that not every gauge theory needs to be ghosted with Fadeev-Popov; for instance, take QED, where no Fadeev-Popov ghosts play any part.

To tackle $\int Dh'$ in (3.27), we introduce two **anti-commuting** variables - ghost variables, ghosts and anti-ghosts. Anti-commuting property of these ghosts make call them as *Grassmann variables* and integrals over them is called *Berezin integral.* So we introduce ghost c^- (c^+) and anti-ghost b_{--} (b_{++}) in determinants as in [43]

$$\begin{aligned}
\det\left(\delta h'_{++}/\delta g\right) &= \int Dc^-(\sigma)Db_{--}(\sigma)\exp\left\{-\frac{1}{\pi}\int d^2\sigma c^-\nabla_+ b_{--}\right\}, \\
\det\left(\delta h'_{--}/\delta g\right) &= \int Dc^+(\sigma)Db_{++}(\sigma)\exp\left\{-\frac{1}{\pi}\int d^2\sigma c^+\nabla_- b_{++}\right\},
\end{aligned} \tag{3.28}$$

and inserting these determinants in (3.27), one get the action with ghosts which solves our problem of the section. But now in our fock space of spectrum states, we have added ghosts and anti-ghosts. We need to differentiate at this junction also. There comes **BRST Formalism**.

BRST quantization is a quantization of BRST charges, however, we only want to know how to differentiate between a state-alike ghost and state itself. In order to that, we need to setup few things. We will consider a Virasoro Algebra, however it will be valid for any Lie Algebra G. We construct some symmetry operators (which forms the representation of G)

$$[K_i, K_j] = f_{ij}^k K_k, \tag{3.29}$$

where f_{ij}^k is structure constant. We now include ghosts and anti-ghosts, c^i and b_i respectively, which follow

$$\left\{b_j, c^i\right\} = \delta_j^i. \tag{3.30}$$

We now have to talk about representation theory of G(and later the cohomology of G). Ghosts transform in the dual of adjoint representation and anti-ghosts transform in the adjoint representation of G. One then can define the ghost number U

$$U = \sum_i c^i b_i. \tag{3.31}$$

We can now say the BRST operator is

$$Q = c^i K_i - \frac{1}{2} f_{ij}{}^k c^i c^j b_k, \tag{3.32}$$

which is a nilpotent of index 2, hence

$$Q^2 = 0. \tag{3.33}$$

We need to find the states which are invariant under this operator; that is, for state Ψ, we can write

$$Q\Psi = 0, \tag{3.34}$$

where Q generates the cohomology of G. It is natural to expect that physical states are cohomology classes of the group. One can now ask about the ghost numbers for which physical states can be found. The most straightforward answer is $U = 0$, because there will be no ghosts and anti-ghosts present. However, it depends on the G.

If one now do this for Virasoro Algebra, the BRST operator is (for open strings[18])

$$Q = \sum_{-\infty}^{\infty} : \left(L_{-m}^{(\alpha)} + \frac{1}{2} L_{-m}^{(c)} - a\delta_m \right) c_m : \tag{3.35}$$

$$U = \sum_{-\infty}^{\infty} : c_{-m} b_m : . \tag{3.36}$$

Of course, Virasoro algebra brings an anomaly as expected. But turns out that it can be easily solved if $D = 26, a = 1$ and $Q^2 = 0$. It also turns out that physical states in the bosonic string theory are BRST cohomology classes of ghost number $-\frac{1}{2}$ [43].

[18]For closed strings, both right and left-moving modes will be added.

3.3 Other Basics

The energy spectrum of on-shell states or physical states is an important area of study. One has theorems like the No-Ghost theorem, which implies the presence of no ghosts in a perfectly stable model. DDF states generate a spectrum. The gain in no-ghosts algebra is now *Lorentz invariance of covariant formalism ensures that no super-conformal ghosts are present in the physical Hilbert space.* DDF operators A^i where i runs over usual transverse numbers. They commute with Virasoro generators

$$[L_m, A^i_n] = 0. \tag{3.37}$$

The states that we gain from light-cone gauge - which are physical, free from bad ghosts, and comes from positive-definite Hilbert space - can be represented in multiples of $SO(D-2)$, however one finds that massive states must be multiples of $SO(D-1)$. For $D = 26$, $SO(24)$ and $SO(25)$ rotate to each other.

String theory classical action can be quantized by - famously - canonical quantization and path-integrals (Polyakov integral). In both means, one finds that algebra for bosonic string theory that is generating when

$$D = 26, \quad a = 1, \tag{3.38}$$

is entirely transverse, free from negative-norms, with many decoupled zero-norms, and is suited for many physical calculations. We call these specific dimensions "Critical dimensions". One can prove it by using the Lorentz-invariance and check if the action is Lorentz (Poincare) invariant at a particular dimension. However, we can have positive spectrum for $D \leq 26$ and $a \geq 1$.

An open string is of two kinds; oriented and unoriented. Basically, an open string has an arrow depicting the direction, and if the endpoints are changed, then the direction is inverted, called an oriented, open string. If it is otherwise, we call it an unoriented open string - similar definitions for closed strings. We have extended Shapiro-Virasoro algebra for oriented closed strings and restricted Shapiro-Virasoro algebra for unoriented closed strings.

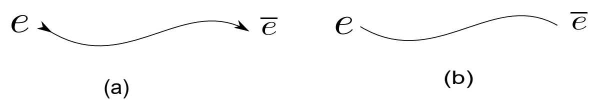

Figure 3.2: Open strings; (a) An oriented open string has distinguished charges at the end (e and $\overline{e}$) and (b) an unoriented open string has no distinguished end charges.

Summary: It, of course, cannot be summed up here. Nonetheless, string theory is just like any other quantum field theory with restrictive algebra for oscillating string-like objects. We find that the most smooth of them is found when $D = 26$ and $a = 1$. The works in string theory, mainly Dual Models and M-theory, have made the field what it is today. It will be exciting to see in the future how this field evolves.

4 Conformal field theory

Conformal field theory is a branch of theoretical physics that studies quantum field theories with a specific type of symmetry called conformal symmetry. Conformal symmetries involve scale transformations (changes in length scales) and special conformal transformations (a combination of translations and inversions). In addition to these transformations, CFTs also respect the usual Poincaré symmetries (translations, rotations, and Lorentz boosts).

CFT gained significant attention around 1984 due to its crucial role in string theory. String theory is a theoretical framework that aims to unify all fundamental forces of nature by considering the fundamental building blocks as one-dimensional "strings" rather than point particles. Conformal symmetry is particularly important in string theory as it helps describe the behavior of strings consistently across different scales.

Conformal field theory is also essential in the field of statistical physics, especially for understanding critical phenomena. Critical phenomena occur near phase transitions, where the behavior of a physical system changes dramatically as it approaches a critical point. CFT provides a powerful tool for describing and solving these critical behaviors, especially in two dimensions. One of the remarkable features of conformal field theory is that some CFTs are solvable exactly in two dimensions. This means that their properties and behavior can be completely determined without approximations, making them valuable tools for studying physical systems in different contexts.

The mention of AdS/CFT correspondence refers to a groundbreaking duality proposed in theoretical physics. It connects two seemingly disparate theories: conformal field theories in lower dimensions and string theories or higher-dimensional gravity theories in anti-de Sitter (AdS) spacetime. This correspondence has opened up new avenues for understanding and solving problems in both theories, and it has sparked renewed interest in the study of conformal field theories.

We shall commence by undertaking a comprehensive examination of the Poincaré group, which encapsulates the symmetries inherent to spacetime. Gradually advancing our discourse, we shall endeavor to augment this group by incorporating the realm of conformal transformations. Subsequently, our attention will be directed toward the meticulous computation of certain correlation functions within the purview of the theory.

4.1 Poincaré algebra

The Poincaré algebra is a fundamental mathematical structure that captures the symmetries of spacetime in special relativity. It provides a mathematical framework for describing the symmetries of translations, rotations, and Lorentz boosts, which are the basic transformations that leave the laws of physics unchanged in special relativity.

The Poincaré algebra is defined by a set of generators and commutation relations. The generators correspond to the various symmetry operations, and the commutation relations describe how these generators interact with each other. In four-dimensional spacetime, the Poincaré algebra has ten generators, which can be organized into four translations, three rotations, and three boosts.

$$x^\mu \to \Lambda^\mu_\nu x^\nu + a^\mu \tag{4.1}$$

where Λ^μ_ν is an element of the Lorentz group of rotations and boosts satisfying

$$\Lambda^\mu_\rho \Lambda^\nu_\sigma \eta_{\mu\nu} = \eta_{\rho\sigma} \tag{4.2}$$

and a^μ is an element of translation subgroup. The general form of an infinitesimal Lorentz transformation in terms of the generators is:

$$\Lambda^\mu_\nu \approx \delta^\mu_\nu - \frac{i}{2}\omega_{\alpha\beta}(J^{\alpha\beta})^\mu_\nu \tag{4.3}$$

where $\omega_{\alpha\beta}$ is the infinitesimal antisymmetric parameter associated with the generators. The $J^{\alpha\beta}$ are the generators of the Lorentz group.

In the context of $o(d-1,1)$ and $O(d-1,1)$, the generators $(J^{\alpha\beta})^\mu_\nu$ are the components of the Lie algebra associated with the Lorentz group $O(d-1,1)$, which is the group of linear transformations that preserve the Minkowski spacetime interval. The generators of the Lorentz group, denoted as $(J^{\alpha\beta})^\mu_\nu$, can be expressed in terms of η as:

$$(J^{\alpha\beta})^\mu_\nu = i(\eta^{\mu\alpha}\delta^\beta_\nu - \eta^{\mu\beta}\delta^\alpha_\nu). \tag{4.4}$$

which satisfies the commutation relation:

$$[J^{\alpha\beta}, J^{\gamma\delta}] = i(\eta^{\alpha\gamma}J^{\beta\delta} - \eta^{\alpha\delta}J^{\beta\gamma} - \eta^{\beta\gamma}J^{\alpha\delta} + \eta^{\beta\delta}J^{\alpha\gamma}). \tag{4.5}$$

These commutation relations reflect the algebraic structure of the Lorentz group, which is essential in the study of spacetime symmetries and the behavior of tensors, vectors, and spinors under Lorentz transformations in special relativity.

In field theory, the generators of the Poincaré group (which combines translations and Lorentz transformations) are realized as differential operators acting on fields. To derive the representation for a scalar field $\phi(x)$, we can start by considering an infinitesimal Lorentz transformation in the active picture.

In the active picture, the Lorentz transformation acts on the fields themselves, as opposed to transforming the coordinates. An infinitesimal Lorentz transformation can be written as:

$$\Lambda^\mu_\nu = \delta^\mu_\nu + i\omega^\mu_\nu, \tag{4.6}$$

where δ is the Kronecker delta and $\omega^{\mu\nu}$ represents the infinitesimal anti-symmetric Lorentz transformation matrix. In the infinitesimal limit, $\omega^{\mu\nu}$ is small, and we can write:

$$\omega^\mu{}_\nu \approx -\frac{1}{2}J^\mu{}_\nu, \tag{4.7}$$

Now, let's derive the representation of the Lorentz generators acting on a scalar field $\phi(x)$. We want to find how $\phi(x)$ changes under an infinitesimal Lorentz transformation:

$$\phi'(x) = \phi(x) + i\omega^{\mu}_{\nu} J^{\mu\nu} \phi(x). \tag{4.8}$$

We can simplify this expression by using the generators you provided earlier:

$$\begin{aligned} \phi'(x) &= \phi(x) + i\omega^{\mu}_{\nu}(i(\eta^{\mu\alpha}\delta^{\beta}_{\nu} - \eta^{\mu\beta}\delta^{\alpha}_{\nu}))\phi(x), \\ \phi'(x) &= \phi(x) + \omega^{\mu}_{\nu}(\eta^{\mu\alpha}\delta^{\beta}_{\nu} - \eta^{\mu\beta}\delta^{\alpha}_{\nu})\partial_{\alpha}\phi(x). \end{aligned} \tag{4.9}$$

Now, let's define the infinitesimal Lorentz transformation operator as an operator acting on $\phi(x)$:

$$\hat{\Lambda}(\omega) = 1 + \omega^{\mu}_{\nu}(\eta^{\mu\alpha}\delta^{\beta}_{\nu} - \eta^{\mu\beta}\delta^{\alpha}_{\nu})\partial_{\alpha}. \tag{4.10}$$

This operator represents an infinitesimal Lorentz transformation. The generator of this transformation is related to the $J^{\mu\nu}$ generators:

$$\hat{J}^{\mu\nu} = i(\eta^{\mu\alpha}\delta^{\beta}_{\nu} - \eta^{\mu\beta}\delta^{\alpha}_{\nu})\partial_{\alpha}. \tag{4.11}$$

In the context of field theory and the Poincaré group, you can define the translation generator, often denoted as P^{μ}, as the differential operator associated with spacetime translations. The action of the translation generator on a scalar field $\phi(x)$ is given by:

$$(P^{\mu}\phi)(x) = -i\partial^{\mu}\phi(x). \tag{4.12}$$

The commutation relations for the Poincaré algebra are:
Translations:

$$[P^{\mu}, P^{\nu}] = 0, \tag{4.13}$$

Lorentz Transformations:

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\sigma}J^{\nu\rho}), \tag{4.14}$$

Mixed Commutation (Translations and Lorentz Transformations):

$$[P^{\mu}, J^{\nu\rho}] = i(\eta^{\mu\nu}P^{\rho} - \eta^{\mu\rho}P^{\nu}), \tag{4.15}$$

4.2 Conformal algebra

4.3 Conformal theories

Theories without scales or dimensionful parameters are classically scale invariant. A simple example is the scalar field with only quartic interaction

$$S = \int \left((\partial\phi)^2 - \frac{\lambda}{4!}\phi^4 \right) d^4x \tag{4.16}$$

In this expression, ϕ represents the scalar field, and λ is a dimensionless coupling constant associated with the quartic interaction term, where the action remains invariant under simultaneous rescaling of spacetime coordinates and the field with specific weights, is known as "conformal invariance" or "scale invariance with a specific weight." In this context, a scale transformation involves multiplying both spacetime coordinates and the field by the same factor, and the requirement of a specific weight indicates how the field's rescaling is related to the rescaling of coordinates.

Space and Time Coordinates Scaling:

$$\begin{aligned} x &\to \lambda x \\ t &\to \lambda t \end{aligned} \tag{4.17}$$

Field Scaling:

$$\phi(x,t) \to \lambda^{\Delta}\phi(\lambda x, \lambda t) \tag{4.18}$$

Here, λ is the scaling factor, and Δ is the specific weight associated with the field's rescaling. For the action to remain unchanged, you need to find the appropriate value of Δ that satisfies this condition.

4.4 Energy-momentum tensor

Noether's theorem relates continuous symmetries of a physical system to conserved quantities. In the context of spacetime translations, the associated conserved current is indeed the energy-momentum tensor, often denoted as $T_{\mu\nu}$. This tensor represents the density and flux of energy and momentum in a system.

The energy-momentum tensor can indeed be defined in terms of the action of a physical theory. Specifically, it is defined as the variation of the action with respect to the metric tensor $g^{\mu\nu}$. The action S is a functional that depends on the spacetime metric $g^{\mu\nu}$ and the fields of the theory:

$$S = \int d^4x \mathcal{L}(g^{\mu\nu}, \phi) \tag{4.19}$$

Here, $\mathcal{L}$ is the Lagrangian density, ϕ represents the fields of the theory, and d^4x is the spacetime volume element.

The energy-momentum tensor $T_{\mu\nu}$ is defined as the functional derivative of the action with respect to the metric:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} \tag{4.20}$$

Here, $\delta S/\delta g^{\mu\nu}$ represents the functional derivative of the action with respect to the metric tensor $g^{\mu\nu}$, and $\sqrt{-g}$ is the determinant of the metric tensor.

The energy-momentum tensor $T_{\mu\nu}$ encodes information about the distribution of energy, momentum, and stress in the system. It has a number of important properties, such as being symmetric ($T_{\mu\nu} = T_{\nu\mu}$) and satisfying the conservation law $\nabla^\mu T_{\mu\nu} = 0$, where ∇^μ is the covariant derivative.

The conservation law $\nabla^\mu T_{\mu\nu} = 0$ is a consequence of Noether's theorem applied to spacetime translation symmetry. It states that the total energy momentum within any closed region of spacetime is conserved, which is a fundamental principle of physics.

We would arrive at the same result by considering the conserved currents for the dilation which is given by:

$$J_{(D)\mu} = x^\nu T_{\mu\nu}, \tag{4.21}$$

This current is conserved, meaning that its divergence $\partial^\nu J_{(D)\mu}$ is zero.

4.5 Correlation functions

In conformal field theory (CFT), correlation functions play a central role as they encode important information about the theory. Conformal symmetry imposes strong constraints on the form of these correlators. Let's consider a two-point correlation function of scalar fields $\phi(x)$ in a CFT:

$$\langle \phi_a(x_1)\phi_b(x_2)\rangle = C_{ab}\frac{1}{|x_1 - x_2|^{\Delta_a+\Delta_b}}, \tag{4.22}$$

where Δ_a and Δ_b are the scaling dimensions of ϕ_a and ϕ_b, respectively, and C_{ab} is a constant that depends on the normalization of the fields. For a non-zero correlation to exist ($C_{ab} \neq 0$), the scaling dimensions of the fields must be equal ($\Delta_a = \Delta_b$). Similarly, consider a three-point correlation function involving three quasi-primary fields $\phi_a(x_1)$, $\phi_b(x_2)$, and $\phi_c(x_3)$:

$$\langle \phi_a(x_1)\phi_b(x_2)\phi_c(x_3)\rangle = C_{abc}\frac{1}{|x_{12}|^{\Delta_a+\Delta_b-\Delta_c}|x_{23}|^{\Delta_b+\Delta_c-\Delta_a}|x_{13}|^{\Delta_c+\Delta_a-\Delta_b}}, \tag{4.23}$$

The two and three-point correlation functions in a conformal field theory (CFT) are indeed highly constrained by conformal symmetry and are essentially fixed, up to overall integration constants, by the algebraic and geometric properties of the CFT. However, the situation changes when considering four-point correlation functions and higher-order correlators. In higher-order correlators, such as four-point functions, conformal symmetry alone is not sufficient to completely determine their form. The reason for this is the emergence of additional degrees of freedom, often referred to as "crossing symmetry" or "crossing relations."

4.6 Ward Identities

The Ward identity associated with translation invariance, which is one of the fundamental spacetime symmetries, is a specific case of Ward identities in conformal field theory (CFT). In this context, translation invariance corresponds to the special case of conformal transformations where only pure translations are considered. The Ward identity for translation invariance relates the correlation functions of conserved currents to the correlation functions of the fields themselves.

The Ward identity for translation invariance is often expressed as follows:

$$\partial_\mu \langle J^\mu(x)\mathcal{O}_1(y_1, y_2, \ldots)\rangle = -i \sum_i \langle \partial_\mu \mathcal{O}_1(y_1, y_2, \ldots)\rangle, \tag{4.24}$$

where, $\mathcal{O}_1(y_1, y_2, \ldots)$ represents a collection of fields or operators in the correlation function. The right-hand side sums over all the fields $\mathcal{O}_1(y_1, y_2, \ldots)$ in the correlation function and computes the derivative of each field with respect to its spacetime coordinate y_i.

5 The Conformal Group in Two Dimensions

The conformal group in two dimensions often denoted $\mathrm{SL}(2, \mathbb{C})$, is a group of coordinate transformations that preserve angles and ratios of infinitesimal distances on a two-dimensional manifold, such as the complex plane. Conformal transformations are particularly important in two-dimensional conformal field theory (CFT) and have numerous applications in physics and mathematics.

5.1 Conformal Mappings

We consider the coordinates (z^0, z^1) on the plane. Let's denote the contravariant metric tensor in the original coordinate system as g^{ij}, and in the new coordinate system as g'^{ij}. The transformation of the contravariant metric tensor under a change of coordinates $z^i \to w^i(x)$ can be expressed as follows:

$$g'^{ij}(x) = \frac{\partial w^i}{\partial z^a}\frac{\partial w^j}{\partial z^b} g^{ab}(z), \tag{5.1}$$

where a and b run over the indices of the original coordinate system, and i and j run over the indices of the new coordinate system. In this context, z typically represents a complex coordinate, and $\bar{z}$ is the complex conjugate of z. This change of coordinates is commonly used in complex analysis and in the study of conformal transformations. The translation rules for these coordinates can be described as follows:

1. Translation from Cartesian to Complex Coordinates:

$$\begin{aligned} z &= z^0 + iz^1 \\ \bar{z} &= z^0 - iz^1 \end{aligned} \tag{5.2}$$

Here, z is a complex number, and $\bar{z}$ is its complex conjugate. This transformation allows you to represent points in the Cartesian (z^0, z^1) coordinates using complex numbers.

2. Translation from Complex to Cartesian Coordinates:

$$
\begin{aligned}
z^0 &= \frac{1}{2}(z + \bar{z}) \\
z^1 &= \frac{1}{2i}(z - \bar{z})
\end{aligned}
\tag{5.3}
$$

These equations allow you to express the complex coordinates z and $\bar{z}$ in terms of the Cartesian coordinates (z^0, z^1).

5.2 Global Conformal Transformations

Global conformal transformations, also known as global conformal mappings, refer to mathematical transformations that preserve angles and distances on a global scale. These transformations are typically used in complex analysis and differential geometry to study how complex functions map one region of the complex plane to another while maintaining the conformal (angle-preserving) property.

In two dimensions (the complex plane), a global conformal transformation is a bijective (one-to-one and onto) mapping $f : \mathbb{C} \to \mathbb{C}$ that is holomorphic (analytic) and satisfies the Cauchy-Riemann equations. Such transformations can be represented by functions of the form:

$$
f(z) = \frac{az + b}{cz + d}, \tag{5.4}
$$

where a, b, c, d are complex constants with $ad - bc \neq 0$.

These transformations are important in various areas of mathematics and physics, including complex analysis, Riemann surfaces, and the study of conformal field theories in theoretical physics.

5.3 Virasoro Algebra

The Virasoro algebra is a specific infinite-dimensional Lie algebra that plays a central role in the study of conformal field theories (CFTs) in theoretical physics, particularly in two-dimensional CFTs. It is closely related to the Witt algebra, as mentioned earlier, but includes an additional central charge term. The Virasoro algebra is named after its discoverer, the physicist Miguel Ángel Virasoro.

The generators of the Virasoro algebra are typically denoted as L_n, where n is an integer. The commutation relations for these generators are given by:

$$
\begin{aligned}
[L_m, L_n] &= (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n,0} \\
[\bar{L}_m, \bar{L}_n] &= (m - n)\bar{L}_{m+n} + \frac{\bar{c}}{12}m(m^2 - 1)\delta_{m+n,0}
\end{aligned}
\tag{5.5}
$$

In this commutation relation, c is called the central charge, and $\delta_{m+n,0}$ is the Kronecker delta function, which equals 1 if $m+n=0$ and 0 otherwise.

The Virasoro algebra contains two key types of generators:

1. **Diffeomorphism Generators (L_n):** These generators are responsible for infinitesimal coordinate transformations or diffeomorphisms on the complex plane in a conformal field theory. They encode how fields transform under scale transformations and reparametrize the spacetime coordinates.

2. **Central Charge (c):** The central charge term is a crucial part of the Virasoro algebra. It gives rise to the so-called "central extension" of the Witt algebra. The value of c is a fundamental parameter of the conformal field theory and affects its properties. The central charge measures the degree of deviation from pure conformal symmetry. Different values of c lead to different types of CFTs, such as minimal models or non-minimal models.

In two-dimensional conformal field theories, the Virasoro generators act on the fields of the theory, and their commutation relations capture the transformations that preserve the conformal symmetry. These generators are associated with the scaling, translation, and reparameterization of spacetime coordinates.

6 Entanglement Entropy

The study of the entanglement content of many-body quantum systems has indeed led to significant insights into these systems, particularly concerning criticality and topological order. When a quantum system is in its ground state or any pure state, understanding the entanglement between its constituent parts can provide crucial information about the system's properties and behavior. This understanding is often quantified using entanglement entropies. Let ρ be the density matrix of a system, which we take to be in a pure quantum state $|\Psi\rangle$ so that $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$. For a composite quantum system consisting of subsystems A and B, the Hilbert space be written as a direct product $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, and we can obtain the density matrix for subsystem A, $\rho_A = \mathrm{Tr}_B(\rho)$. The entanglement entropy is the corresponding von Neumann entropy

$$S_A = -\mathrm{Tr}(\rho_A \log(\rho_A)) \tag{6.1}$$

When ρ corresponds to a pure quantum state $S_A = S_B$. Other standard measures of bipartite entanglement in pure states are the Renyi entropies:

$$S^{(n)}(\rho_A) = \frac{1}{1-n} \log\left(\mathrm{Tr}(\rho_A^n)\right), \tag{6.2}$$

that also satisfy $S_A^{(n)} = S_B^{(n)}$ whenever ρ corresponds to a pure quantum state. When n is close to 1, the Rényi entropy gives similar information to the von Neumann entropy and captures the dominant features of the entanglement structure. For $n > 1$, it emphasizes the contributions of the more entangled states, while for $n < 1$, it is more sensitive to less entangled states.

7 Random Matrix Theory

Random matrix theory (RMT) is a classic example of statistical group theory in general physics. The most recent development for RMT is the equivalence of JT gravity with RMT [54]. From the correspondence of AdS/CFT, one learned that a bulk theory with gravity lives on the boundary of a quantum system. However, the equivalence of JT gravity is not given to a boundary theory (or a bulk theory). In fact, RMT shares the correspondence with JT gravity; hence JT gravity is a dual to random matrix integral of Hamiltonian H, where H is a random matrix.

JT gravity [55] is not part of the present discussion of the paper. However, a very brief summary follows. String theory aims to understand gravity in higher dimensions. Kaluza-Klein theory tries for five dimensions, but how would gravity theories look in 2 dimensions? That is answered by JT gravity which operates gravity in 2 dimensions. Recently, many developments have been made in JT gravity, especially its relation to a random matrix integral [54, 56, 57].

RMT is mainly concerned about the statistics of groups, at least for us, whose applications are wide in physics. It is a fascinating fact that RMT is not only for physics but for many other subjects[19].

Consider a matrix M, from linear algebra we know that M holds eigenvalues m_{ij} (for 2×2 matrix). Suppose the elements of matrix M are random variables, to which we say to obey some specific outlined properties. In that case, the study of those (m_{ij}) eigenvalues is called the "random matrix theory" problem. Now one can ask what the practical application of RMT is. Actually, there are many practical applications, consider the well-studied example of the nucleus using these random matrices, which was developed by Wigner (and Dyson). And the recent example of the success of RMT is JT gravity. We suggest the reader to [58].

The most interesting and simple case of the matrix is Hamiltonian. We consider a Hamiltonian H of a system[20]. From matrix theory, H is also a matrix and random variables of H_{ij} yield random eigenvalues E_{ij}. Hamiltonian matrices are normal at low-energy levels, but finding the numbers becomes a hard task with precision at high-energy levels, so we say that those numbers under some symmetry requirement are random variables and are subject to random matrices. One can immediately realize that random variables are the perfect key of chaotic systems (see [59]), where a minor change in the initial condition results in a major change in the final condition. Back to the subject, why do we need to understand random variables of H in this case?

The answer is trivial and supported by the fact of statistics. When we study any RMT - we basically - try to understand how the system would look under random changes or random variations. These random changes are not wholly arbitrary but are confined under some specific outline. These random matrices can be types of orthogonal matrices, symplectic matrices, or unitary matrices. They can either obey 'Gaussian distribution' or 'circular distribution.'

A random matrix can be from any one of 10 classes. These classes are also called ensembles. Among 10 ensembles, three are Dyson ensembles and seven are Artland-Zirnbauer ensembles [60].

[19]Like sociology.

[20]In the primitive examples, the system was a nuclear theory.

Dyson ensembles are simple symmetry ensembles of Gaussian nature. Three Dyson ensembles are Gaussian Orthogonal Ensemble (GOE), Gaussian Unitary Ensemble (GUE), and Gaussian symplectic Ensemble (GSE). In the original theory of random matrices in nuclear physics if Hamiltonian H is assumed without time-reversal symmetry, then the random theory is of GUE. (Anti-unitary is linked to time-reversal). If the time-reversal is assumed, then the simple that happens is that $U(n)$ reduces to subgroup $O(n)$ if $\mathsf{T}^2 = 1$ and $Sp(n)$ if $\mathsf{T}^2 = -1$. Where H^i_j are hermitian random matrices and i, j runs over n. $O(n)$ matrices are symmetric ($H_{ij} = H_{ji}$) and $Sp(n)$ matrices take the form $H^i_j = \epsilon^{ik} H^k_j$, where ϵ^{ik} is anti-symmetric tensor. The other seven Artland-Zirnbauer classes are divided into two parts;

1. For the 4 cases, we have covariant (or contravariant) indices H_{ij}. If the group is $U(n)$, then H_{ij} can either be a symmetric tensor or anti-symmetric tensor. If $O(n)$ is considered, then H can be an anti-symmetric and for $Sp(n)$, it will be a symmetric tensor.
2. The last 3 cases are for product groups $O(n) \times O(n), U(n) \times U(n)$ or $Sp(n) \times Sp(n)$. We can also introduce a number c such that the product group is $O(n) \times O(n + c)$ and so on for different product groups; in doing so, we make our H to be bifundamental where one index will transform under the first group and another index under the second one.

Random matrices have become quite active in field theory calculations, mainly in condensed matter calculations. The broader aspect of random matrices can be experienced in models like SYK (Sachdev-Yao-Kitaev), JT gravity, and Black Holes [61]. It is a game of patience to see where random matrix theory proceed.

A Anti-De Sitter Space

Anti-De Sitter (AdS) is a geometrical solution to Einstein's equation of general relativity. Its counterpart is de-Sitter space (dS), which is actually a higher-dimensional positive scalar curved manifold. Positive scalar curvature indicates that the Λ (cosmological constant) is positive. We represent dS space with dS_n, where n implies that dS is Lorentzian analogue of n spheres. Moreover, it is also the solution to general relativity (GR), which is a maximally symmetric theory. Furthermore, anti-de Sitter space is the anti-analog of de-Sitter. Anti-de Sitter has negative scalar curvature ($\Lambda < 0$).

AdS are of high interest in theoretical physics, see for example [62, 63], which showed that AdS is a theory living on the boundary of the conformal theory. This is what we called "AdS/CFT Correspondence". It is in contrast to JT gravity and SYK models, discussed in section 7, which are dual to ensembles.

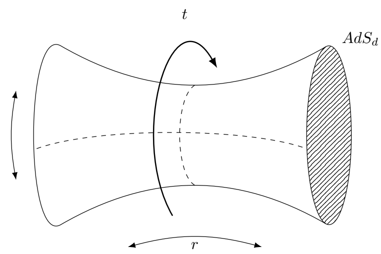

Figure A.1: The topology of AdS_d as embedded into flat space-time of d+1 dimensions.

A.1 Geometry

As we noted out that AdS is a solution to GR, then it must be rich in geometry. The geometry of AdS is too topologically relativistic at first sight. We first observe that the solution to GR action, in this case, is given by

$$S = \frac{1}{16\pi G_N}\int d^{d+2}x\sqrt{-g}(R-2\Lambda), \tag{A.1}$$

$$= \frac{1}{16\pi G_N}\int d^{d+2}x\sqrt{-g}\left(R+\frac{(d+1)d}{L^2}\right), \tag{A.2}$$

which further lead to the solution for the cosmological constant;

$$R_{\mu\nu} = -\frac{d+1}{L^2}g_{\mu\nu}, \tag{A.3}$$

$$R = -\frac{(d+1)(d+2)}{L^2}, \tag{A.4}$$

$$\Lambda = -\frac{(d+1)d}{2L^2}. \tag{A.5}$$

Moreover, AdS can be defined for $d \geq 2$. The geometry of AdS can be compacted as a hypersurface[21] into a Euclidean space. For instance, a 4d AdS is a hypersurface in a 5d flat space with three spatial and two-time dimensions, where the AdS has three spatial and one time dimensions. Fig. (A.1) closely captures such topology. So, AdS can be seen as a Lorentzian analog of a sphere that can be visualized as embedded into a flat space-time. Note that the extra dimension is timelike, not spacelike. For $\mathbb{R}^{3,2}$, we write the metric as (in $-,-,+,+,+$)

$$ds^2 = -dP^2 - dQ^2 + dX^2 + dY^2 + dZ^2, \tag{A.6}$$

where P and Q are timelike dimensions and X, Y and Z are spacelike dimensions. AdS follow this immediately for 4d hypersurface in $\mathbb{R}^{3,2}$, which is given by

$$-P^2 - Q^2 + X^2 + Y^2 + Z^2 = -a^2, \tag{A.7}$$

[21] Hypersurface is a manifold that generalizes the hyperplane and surface. It can be seen as an embedded geometry into some Euclidean space or, for that matter, the affine-connected spaces. A "d" dimensional AdS exists in "$d+1$" plane. (A hypersurface is also a sub-manifold of codimension 1.)

Figure A.2: Penrose diagram of AdS, where time will be on boundary.

where a is the curvature scale of AdS and has dimensions of length. Eq. (A.7) is also dubbed as an equation for *hyperboloid*.[22] When there are no time dimensions at all, then the anti-de sitter space shows complete hyperboloid behavior. If the timelike dimension is one, then it shows the Lorentzian signature (quasi-sphere).

Another point to note is that the Penrose diagram for AdS is a cylinder - fig A.1 - so for an observer along the geodesic line, a particle can go to boundary and come back in *finite* time, see [64] where it has been described using causal paths and Penrose diagrams.

The isometry group of AdS_4 space is $SO(4,2)$. And the isometries of AdS are in one-to-one correspondence with the generators of the conformal group; that is another way of saying AdS is dual to CFT.

A.2 Duality

We have mentioned the duality several times in the paper, but what exactly, however in vague terms, this duality means? (For full description, we refer to [15].) String theory is categorized into five parts, namely Type I, Type IIA, and Type IIB, 11D Supergravity, $SO(32)$ Heterotic, and $E_8 \times E_8$ Heterotic. And the simplest correspondence one can take is $AdS_5 \times S_5$ and $N = 4$ SYM, where string theory IIB is compactified on $AdS_5 \times S_5$. Important idea to stress is, what we say, that $N = 4$ SYM lives on the boundary of $AdS_5 \times S_5$, however, IIB lives on $AdS_5 \times S_5$.

B $T\bar{T}$ Deformations and Bulk

In this little disconnected appendix, we would review some of the latest developments in $T\bar{T}$ deformations (or T^2 in $d > 2$).

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[22] A hyperboloid is a surface obtained by the rotation of hyperbola.

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