

CHAPTER 7

Topos and Sheaf Theory

“L’état de vérité partiel serait alors l’état de silence intérieur et d’écoute.”

– Alexander Grothendieck in La Clef des Songes

“I do not approve. I understand.” – Mr. Spock to Anan 7

7.1. In the previous chapter, our intention was to introduce a few objects and mathematical notions like that of gluing (for example that of projective space from affine charts), local rings, and coordinate rings. In this chapter, our goal is to introduce the notion of topological spaces, shaves, ringed spaces and topoi which make up interesting part of EGA and SGA.

For reasons, we will bi-fold this chapter into Part I which will be about the local-to-global language of algebraic geometry and Part II which will the categorical viewpoint towards the notion of space.

Part I: The local-to-global language

7.2. The best point to start this local-to-global language is the *sheaf theory* which was precisely developed by Leray¹ [32] for analyzing the obstruction of piecing together local data to get global data.

1. Presheaf and Sheaf

DEFINITION 7.3. Let X be a topological space. We define presheaf $\mathcal{F}$ on X with the assignment of $\mathcal{F}(U)$ to each open set $U \subset X$ such that for every inclusion of opens $V \subset U$, we have restriction map

$$\text{res}_U^V : \mathcal{F}(U) \rightarrow \mathcal{F}(V) \quad (195)$$

such that

This is a part of Prelude of Schemes project available at <https://aayushayh.github.io/PtoS.html>.
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¹It is also interesting to note that Leray developed this 1946 while he was a prisoner of war in Oflag XVII-A in Austria during World War II.

- (1) $\text{res}_U^U = \text{id}_{\mathcal{F}(U)}$ for all the opens U
- (2) Whenever $W \subset V \subset U$, we have

$$\text{res}_V^W \cdot \text{res}_U^V = \text{res}_U^W \quad (196)$$

which is equivalent to saying that the following commute

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\quad} & \mathcal{F}(W) \\ & \searrow & \nearrow \\ & \mathcal{F}(V) & \end{array}$$

DEFINITION 7.4 (Morphism of presheaves). A morphism of presheaves $\mathcal{F}, \mathcal{G}$ on X is a collection of maps $\alpha : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ for each open open set on X such that whenever there is inclusion of opens sets, the following commutes

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\alpha(U)} & \mathcal{G}(U) \\ \text{res}_U^V \downarrow & & \downarrow \text{res}_U^V \\ \mathcal{F}(V) & \xrightarrow{\alpha(V)} & \mathcal{G}(V) \end{array}$$

whenever $V \subset U$.

7.5. In many places, one defines an element f in $\mathcal{F}(U)$ to be the *section* over U . We can further simply $\text{res}_U^V(f)$ as $\sigma|_V$.

DEFINITION 7.6 (Presheaf as a functor). In the language of categories, we can define a presheaf (of sets) on $\mathcal{C}$ to be a ‘contravariant’ functor from the category $\mathcal{C}$ (in our previous examples, $\mathcal{C}$ is the category of open subsets of a fixed space) to the category of sets (again, it can be any other category).

The category of presheaves (of sets) is denoted by $PSh(\mathcal{C})_{Sets}$. The morphisms in this category are the natural transformations mentioned in Def. 7.4.

7.7 (EGA1 3.1.5). If we have a morphism of sheaf $\alpha : \mathcal{F} \rightarrow \mathcal{G}$ on X , then for $U \subset X$, there is a morphism defined on the level of $\alpha_U : \mathcal{F}|_U \rightarrow \mathcal{G}|_U$.

In particular, if we have two sheaves $\mathcal{F}, \mathcal{G}$ on X , then we can define the ‘sheaf $\mathcal{H}om$ ’ the sheaf

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) = \text{Hom}(\mathcal{F}|_U, \mathcal{G}|_U) \quad (197)$$

where $\text{Hom}(\mathcal{F}, \mathcal{G})$ is the set of morphism.

REMARK 7.8. Since $\mathcal{F}$ is a contravariant functor, the morphism of sheaves is a natural transformation.

Note that the category of the presheaves on $\mathcal{C}$ the functor category², denoted as $PSh(\mathcal{C})$. The category of sheaves $Sh(\mathcal{C})$ on $\mathcal{C}$ is the full subcategory of $PSh(\mathcal{C})$.

DEFINITION 7.9 (Germ). Let $U, V \subset X$ and $\mathcal{F}$ be a presheaf. If we have two sections $f \in \mathcal{F}(U)$ and $g \in \mathcal{F}(V)$, and if they agree to an open neighbourhood $W \subseteq U \cap V$ through

$$f|_W = g|_W \quad (198)$$

then they are called germs.

In other words, two pairs (f, U) and (g, V) are equivalent are called germ at $p \in X$ if they satisfy (198) for an open neighborhood $W \subseteq U \cap V$ of a point p .

DEFINITION 7.10 (Stalks). A stalk $\mathcal{F}_p$ at $p \in X$ is the collection of germs at p .

7.11. A very interesting method of studying a presheaf $\mathcal{F}$ is by studying the sections over smaller and smaller neighborhoods of a point $x \in X$.

DEFINITION 7.12 (One more definition of Stalks). We define the stalk at point x as $\mathcal{F}_x$ to be $\varinjlim_{x \in U} \mathcal{F}(U)$ for $x \in U$ where U 's are the open neighborhood of x which are partially ordered by the inclusion³.

7.13. Given a morphism of presheaves $\alpha : \mathcal{F} \rightarrow \mathcal{G}$, there is a family of morphisms of stalks $\alpha_x : \mathcal{F}_x \rightarrow \mathcal{G}_x, \forall x \in X$.

DEFINITION 7.14 (A random but important definition here of étale space). Let X be a topological space and $\mathcal{F}$ be a sheaf of sets over X . We define the *étalé space* of $\mathcal{F}$ as

$$E(\mathcal{F}) = \coprod_{x \in X} \mathcal{F}_x \quad (199)$$

with a projection $p : E(\mathcal{F}) \rightarrow X, (x, s_x) \mapsto x$ where s_x is the germ at x and the topology is given by subsets $\tilde{s}(U) := \{(x, s_x) : x \in U\}$ where $U \subset X$ are open and s is a section in $\mathcal{F}(U)$.

The projection is a local homeomorphism.

7.15. We can construct the sheaf from the continuous sections of $p : E(\mathcal{F}) \rightarrow X$ over open subsets of X .

The point that the sheaf as a topological space assembled from its stalks is given by above definition. However, as mentioned in EGA 3.1.6, one does not work with sheaf as a topological space when (the sheaf on a category) $\mathcal{C}$ has additional structures on morphism.

²A functor category $[C, \mathcal{A}]$ is collection of functors $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{A}$ and the morphisms are natural transformations between functors.

³The philosophy is that $\mathcal{F}_x$ is the disjoint union $\bigcup \mathcal{F}(U)$ of all the sections of $\mathcal{F}$ modulo equivalent sections (germs).

Bibliography

- [1] Bourbaki, *Algèbre*. Springer, 2007.
- [2] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*. CRC Press, 2018.
- [3] R. Bott and L. W. Tu, *Differential forms in algebraic topology*, vol. 82. Springer Science & Business Media, 2013.
- [4] D. A. Buchsbaum, “Exact categories and duality,” *Transactions of the American Mathematical Society* **80** no. 1, (1955) 1–34.
- [5] A. Grothendieck, “Sur quelques points d’algèbre homologique,” *Tohoku Mathematical Journal, Second Series* **9** no. 2, (1957) 119–183. English translation available at <https://www.math.mcgill.ca/barr/papers/gk.pdf>.
- [6] F. Peter, “Abelian Categories: An Introduction to the Theory of Functors,”.
- [7] J.-L. Verdier, “Catégories dérivées Quelques résultats (état 0),” in *Cohomologie Etale: Séminaire de Géométrie Algébrique du Bois-Marie SGA 4 1/2*, pp. 262–311. Springer, 1977.
- [8] C. A. Weibel, *An introduction to homological algebra*. No. 38. Cambridge university press, 1994.
- [9] D. Happel, *Triangulated categories in the representation theory of finite dimensional algebras*, vol. 119. Cambridge University Press, 1988.
- [10] S. MacLane, “Henri Cartan and Samuel Eilenberg, Homological Algebra,”.
- [11] T. Stacks project authors, “The Stacks project.” <https://stacks.math.columbia.edu>, 2025.
- [12] C. A. Weibel, *The K-book: An Introduction to Algebraic K-theory*, vol. 145. American Mathematical Soc., 2013.
- [13] M. Atiyah, *K-theory*. CRC press, 2018.
- [14] H. Miller, “Leray in Oflag XVIII: The origins of sheaf theory, sheaf cohomology, and spectral sequences,”.
- [15] J.-P. Serre, “Faisceaux algébriques cohérents,” *Annals of Mathematics* **61** no. 2, (1955) 197–278.
- [16] A. Grothendieck, “Éléments de géométrie algébrique: I. Le langage des schémas,” *Publications Mathématiques de l’IHÉS* **4** (1960) 5–228.
- [17] A. Verma, “Local aspects of topological quantization and the Wu-Yang Monopoles,” *arXiv:2406.18799* (2024) .
- [18] T. Y. Chow, “You could have invented spectral sequences,” *Notices of the AMS* **53** (2006) 15–19.
- [19] A. HATCHER, “Algebraic Topology,” <http://www.math.cornell.edu/~hatcher/AT/ATpage.html> (2002) .
- [20] J. McCleary, *A user’s guide to spectral sequences*. No. 58. Cambridge University Press, 2001.
- [21] M. Dubois-Violette, “The Weil-BRS algebra of a Lie algebra and the anomalous terms in gauge theory,” *Journal of Geometry and Physics* **3** no. 4, (1986) 525–565.
- [22] R. Vakil, “The rising sea: Foundations of algebraic geometry,”. available at <https://math.stanford.edu/~vakil/216blog/FOAGjul2724public.pdf>.
- [23] W. Browder, “Torsion in H-spaces,” *Annals of Mathematics* **74** no. 1, (1961) 24–51.
- [24] A. Weil, *Foundations of algebraic geometry*, vol. 29. American Mathematical Soc., 1946.

BIBLIOGRAPHY

- [25] R. Hartshorne, *Algebraic geometry*, vol. 52. Springer Science & Business Media, 2013.
- [26] D. Mumford, *The red book of varieties and schemes: includes the Michigan lectures (1974) on curves and their Jacobians*, vol. 1358. Springer, 2004.
- [27] D. Dummit and R. Foote, *Abstract Algebra*. Wiley, 2003.
- [28] D. Hilbert, “Über die vollen Invariantensysteme,” in *Algebra• Invariantentheorie Geometrie*, pp. 287–344. Springer, 1970.
- [29] H. Matsumura, *Commutative ring theory*. No. 8. Cambridge university press, 1989.
- [30] C. Birkar, “Lectures on birational geometry,” *arXiv preprint arXiv:1210.2670* (2012) .
- [31] J.-P. Serre, “Géométrie algébrique et géométrie analytique,” in *Annales de l’institut Fourier*, vol. 6, pp. 1–42. 1956.
- [32] J. Leray, “Lanneau dhomologie dune representation,” *CR Acad. Sci. Paris* **222** (1946) 13661368.