CHAPTER 5

Homological Algebra

> "For my own sake, I have made a systematic (as yet unfinished) review of my ideas of homological algebra. I find it very agreeable to stick all sorts of things, which are not much fun when taken individually, together under the heading of derived functors."
> - Alexander Grothendieck (in his letter to Serre)

5.1. Homological algebra was the most important development by Grothendieck, for example in his Tohuku paper. Our goal in this chapter is to discuss homological algebra in the context of EGA seminars. Of course, some of the material here might be very familiar to you. We will discuss the abstract idea behind chain complexes and them being valued in any category. We will introduce homotopy and chain maps in this chapter. Our focus will be to introduce the homotopy category of chain complexes and do some basic discussions about derived categories as well. Derived functors are discussed as well[1].

1. Short Exact Sequences and Chain Complexes

5.2. Most of the focus on abelian categories are given because they admit short exact sequences

$$0 \xrightarrow{f_1} A \xrightarrow{f_2} B \xrightarrow{f_3} C \to 0 \tag{3}$$

such that at each point, it is 'exact' which means that $ker(f_{n+1}) = im(f_n)$ and also notice that $f_{n+1} \circ f_n = 0$.

5.3. With the above definition, we have following examples

(1) A sequence $0 \to V \xrightarrow{f} W$ is exact if f is injective.
(2) A sequence $V \xrightarrow{f} W \to 0$ is exact if f is surjective.
(3) A sequence $0 \to V \xrightarrow{f} W \to 0$ is exact if f is an isomorphism.

[1]A good part of these notes are taken from my written notes (and a poster) in a fall 2025 course in representation theory of quivers by Prof. Amit Kuber.

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(4) A sequence $0 \to V \xrightarrow{f} W \xrightarrow{g} U \to 0$ is exact if f is injective, g is surjective and $ker\ g = im\ f$.

(5) Given $V \xrightarrow{f} W$, the following is always exact

$$0 \to ker\ f \xrightarrow{i} V \xrightarrow{f} W \xrightarrow{q} cok\ f \to 0 \tag{4}$$

then $ker\ q = im\ f$ and $ker\ f = im\ i$.

Definition 5.4. A chain complex is defined as a sequence

$$\cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots \tag{5}$$

where the $f_n \circ f_{n+1} = 0$. The maps d_n are called differentials.

For example, for $R - Mod$, it is a family of $R - Mod$, $\{C_n\}_{n\in\mathbb{Z}}$ where $C_n \in R - Mod$.

5.5. We can define the kernel of d_n to be the module of n-cycles in $R - Mod$, denoted by $Z_n(C_\bullet)$. Again, we can define image of d_n to be the module of n-boundaries in $R - Mod$, denoted by $B_n(C_\bullet)$. It is clear to see that

$$0 \subseteq Z_n \subset B_n \subset C_n \tag{6}$$

Alternatively, $ker(f_{n+1}) \subseteq img(f_n)$.

5.6. For a chain complex to be exact, we just mention that $Z_n = B_n$ for some chain complex $C_\bullet$.

Definition 5.7. When a chain complex fails to be exact, which means that Z_n/B_n is non-trivial quotient group. This guotient group is called 'Homology' $H_n(C_\bullet)$.

This essentially measures by how much the chain complex fails to be 'exact' at n.

5.8. Similarly, one can define cochain complexes and cohomology groups for them. These are dual descriptions.

Example 5.9. Now we will look at a very interesting example of . Given V, W in $Vect_k$, the sequence

$$0 \to V \xrightarrow{i_1} V \oplus W \xrightarrow{\pi_2} W \to 0 \tag{7}$$

where $i_1 : V \to V \oplus W$ is an inclusion map and $\pi_2 : V \oplus W \to W$ is a projection map, the above sequence (7) is an exact sequence since $ker(\pi_2) = im(i_1)$. This sequence will serve as a canonical example of split sequences.

Definition 5.10 (Split Sequence). A short exact sequence

$$0 \to V \xrightarrow{f} W \xrightarrow{g} U \to 0 \tag{8}$$

is said to be split if $\exists\ g' : U \to W$ such that $g \circ g' = 1_U$

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r] & W \arrow[r, shift left, "g"] & U \arrow[l, shift left, dashed, "g'"] \arrow[r] & 0 \end{tikzcd}$$

For example, the below sequence splits

$$0 \longrightarrow V \xrightarrow{i_1} V \oplus W \underset{i_2}{\overset{\pi_2}{\rightleftarrows}} W \longrightarrow 0$$

We have the following proposition.

5.11. A short exact sequence

$$0 \to V \xrightarrow{f} W \xrightarrow{g} U \to 0 \tag{9}$$

is said to be split if $\exists\ f' : W \to V$ such that $f' \circ f = 1_V$. So both are equal definitions which means that for our example, the existence of i_2 implies the existence of π_1. These are statements of left split and right split in the Splitting Lemma. In general, the existence of g' will imply that f' exists. To see if this is true, we can look at the following diagram

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r, "f"] \arrow[d] & W \arrow[r, "g"] \arrow[d, shift right, "\binom{f'}{g}"'] & U \arrow[r] & 0 \\
0 \arrow[r] & V \arrow[r, "i_1"] & V \oplus U \arrow[r, "\pi_2"] \arrow[u, shift right, "(f\ g')"'] & U \arrow[r] & 0
\end{tikzcd}$$

where $(f\ g')$ is an isomorphism and the diagram must commute.

We suggest that the

$$0 \to V \xrightarrow{i_1} V \oplus W \xrightarrow{\pi_2} W \to 0 \tag{10}$$

is the canonical example of a split sequence. Any **s.e.s.** which is isomorphic to above sequence (10) is also a split sequence.

2. Chain Complexes and Chain Maps

DEFINITION 5.12. We define a category of chain complexes $Ch(\mathcal{A})$ with objects as chain complexes, where the morphism are *chain complex maps*. Given two chain complexes $C_\bullet, D_\bullet$, we have a chain map u

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r] \arrow[d, dashed, "u_{n+1}"] & C_n \arrow[r] \arrow[d, dashed, "u_n"] & C_{n-1} \arrow[r] \arrow[d, dashed, "u_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r] & D_n \arrow[r] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

where each square commutes. It is an important fact that a chain map $u : C_\bullet \to D_\bullet$ induces a map $H_n(C_\bullet) \to H_n(D_\bullet)$ since u sends boundaries $B_n(C_\bullet)$ to boundaries $B_n(D_\bullet)$ and cycles $Z_n(C_\bullet)$ to cycles $Z_n(D_\bullet)$.

Definition 5.13. We define *splitting maps* s_n for some chain complex C_n as $s_n : C_n \to C_{n+1}$ such that it is called a split chain complex if $d_n = d_{n+1} \circ s_n \circ d_n$. If given two chain complexes C_n and D_n with a chain map $u_n : C_n \to D_n$, we choose maps $s_n : C_n \to D_{n+1}$

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "d^C_{n+1}"] \arrow[d, "u_{n+1}"'] & C_n \arrow[r, "d^C_n"] \arrow[d, "u_n"] \arrow[dl, "s_n"'] & C_{n-1} \arrow[r, "d^C_{n-1}"] \arrow[d, "u_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "d^D_{n+1}"] & D_n \arrow[r, "d^D_n"] & D_{n-1} \arrow[r, "d^D_{n-1}"] & \cdots
\end{tikzcd}$$

where the commutativity is given by $d^D_{n+1} \circ u_{n+1} = u_n \circ d^C_{n+1}$ and the chain map $u_n = d_{n+1} \circ s_n + s_{n-1} \circ d^C_n$.

5.14. When there exist splitting maps $s_n : C_n \to D_{n+1}$ and a chain map $u_n : C_n \to D_n$ where $u_n = d_{n+1} \circ s_n + s_{n-1} \circ d^C_n$, then u is called a null homotopic chain map. When given two chain maps $u_n, v_n : C_n \to D_n$, we call them *null homotopic* if their difference is

$$u_n - v_n = d_{n+1} \circ s_n + s_{n-1} \circ d^C_n \tag{11}$$

and the maps $\{s_n\}$ are called chain homotopy from u to v.

5.15. Furthermore, a chain map $u_n : C_n \to D_n$ is a homotopy equivalence if there exists a map $v_n : D_n \to C_n$ such that uv is chain homotopic to the identity on D and vu is chain homotopic to the identity on C.

3. Snake's Lemma

5.16. Snake's Lemma is a powerful tool to create six term exact sequences from two short exact sequences with a zero object. The proof of the lemma is usually a fun diagram chase. A succinct proof is also available in the beginning of the movie *It's my turn, 1980.* We will focus on the heuristics of the lemma.

5.17. Given two row short exact sequences in an abelian category $\mathcal{A}$, when we have a commutative diagram

$$\begin{tikzcd}
 & A \arrow[r, "a"] \arrow[d, "f"'] & B \arrow[r, "b"] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C' &
\end{tikzcd}$$

it gives us a six-term exact sequence

$$ker(f) \longrightarrow ker(g) \longrightarrow ker(h) \xrightarrow{\partial} coker(f) \longrightarrow coker(g) \longrightarrow coker(h)$$

where $\partial : ker(h) \to coker(f)$ is a connecting homomorphism. If the morphism a is a monomorphism, then $ker(f) \to ker(g)$ is a monomorphism and if the morphism b' is an epimorphism, then so is $coker(g) \to coker(h)$.

We obtain the exact sequence by expanding the commutative diagram which gives us a sequence in shape of a 'slithering snake' [A visual diagram of the Snake!]

$$\begin{tikzcd}
 & & ker(f) \arrow[d] & \cancel{ker(g)} \arrow[d] & ker(h) \arrow[d] \arrow[rr, dashed] & & \bullet \arrow[d] \\
\bullet \arrow[dd, dashed] & & A \arrow[r, "a"] \arrow[d, "f"] & B \arrow[r, "b"] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 & \bullet \arrow[llllll, dashed] \\
 & 0 \arrow[r] & A' \arrow[r, "a'"] \arrow[d] & B' \arrow[r, "b'"] \arrow[d] & C' \arrow[d] & & \\
\bullet \arrow[rrrr, dashed] & & \cancel{coker(f)} & \cancel{coker(g)} & coker(h) & &
\end{tikzcd}$$

The connecting homomorphisms ∂ are important in constructing long exact sequences in homological algebra. The proof of the lemma is given by 'diagram chasing'. The proof requires two step: 1) constructing the connecting homomorphism and 2) proving exactness at each point.

A detailed proof will be added in second draft of these notes (updated 05-Nov-25).

4. Abelian Categories and Homotopy Category of Complexes

Definition 5.18 (Pre-Additive Category). A category $\mathcal{C}$ is pre-additive if each morphism set $Mor_{\mathcal{C}}(a, b)$ has the structure of an abelian group such that the composition

$$Mor(a, b) \times Mor(b, c) \to Mor(a, c) \tag{12}$$

is bilinear.

Definition 5.19. An object which is both a final object and an initial object in a pre-additive category $\mathcal{C}$ is called a zero object and denoted by 0.

Definition 5.20 (Additive Category). We call a pre-additive category $\mathcal{C}$ *additive* category if it admits finite bi-products[2].

Definition 5.21 (Abelian Category). We call a category $\mathcal{C}$ abelian if it satisfies the following properties

(1) It is an additive category.

[2]The finite products and fintie co-products coincide here

(2) Kernels and their dual co-kernels exist.
(3) Every injective morphism is a kernel of its own co-kernel.
(4) Every surjective morphism is a co-kernel of its own kernel.

5.22. Essentially, an abelian category is an abstraction of the (basic) properties of category of abelian groups. It is believed to be introduced by Buchsbaum [**3**] in 1955 as exact categories and later standardized by Grothendieck in his Tohoku paper [**4**] using axiomatic approach. However, the term 'abelian category' was termed by Freyd [**5**].

Example 5.23. A good example of abelian category is category of representation of a group $Rep(G)$ or the module category of an artinian algebra Λ over field k.

5.24. The most interesting reason for studying abelian categories is that they admit short exact sequences that we have discussed above. In later parts of this project, we will be interested to read more than exact sequences like *triangles* in triangulated categories.

5.25. We will digress to recall what is a quasi-isomorphism. When we want to just know the homological information of the complexes and their equivalence.

Definition 5.26. A chain map of complexes $u : C_\bullet \to D_\bullet$ in abelian category $\mathcal{A}$ is called a quasi-isomorphism if the induced homology morphism $u_* = H^n(C_\bullet) \to H^n(D_\bullet)$ is an isomorphism.

From homology point of view, two complexes $C_\bullet$ and $D_\bullet$ become distinguishable.

Example 5.27. For example, there exists a quasi-isomorphism among any two projective resolutions (or injective resolutions) of same object.

Theorem 5.28. If a map $u : C_\bullet \to D_\bullet$ is a homotopy equivalence, then it is a quasi-isomorphism.

5.29. Note that a quasi-isomorphism is not always a homotopy equivalence.

Definition 5.30. Let $\mathcal{A}$ be an additive category, we define a homotopy category of chain complexes $K(\mathcal{A})$ as follows

- Objects: The objects of $K(\mathcal{A})$ are chain complexes.
- Morphisms: For any two chain complexes $C_\bullet, D_\bullet \in \mathcal{A}$, we define the set of morphisms as the set of homotopy classes of chain maps from $C_\bullet$ to $D_\bullet$

$$Hom_{K(\mathcal{A})}(C_\bullet, D_\bullet) = Hom_{Ch(\mathcal{A})}(C_\bullet, D_\bullet)/\sim \tag{13}$$

where any two chain maps u, v are chain homotopic, then $u \sim v$.

5.31. Essentially, we can define a functor $Ch(\mathcal{A}) \to K(\mathcal{A})$ which is identity on the objects and quotient projection on the morphisms. Moreover, $K(\mathcal{A})$ is an additive category as well as *a triangulated category*, however, it is not an abelian category in general.

5.32. The rescue needed to do homological algebras for $K(\mathcal{A})$, since it fails to be abelian category, is provided by triangulated structure on it which contain distinguished triangles from which we get a long exact sequence of homology groups. While $K(\mathcal{A})$ is a 'nice' category to work with homotopical settings. It fails to identify the quasi-isomorphisms and that motivates the construction of derived categories.

5.33. The issue with $K(\mathcal{A})$ being that it does not recognize homotopy equivalence which are not quasi-isomorphism requires an abstract settings of 'derived categories'. The goal is then to construct a category where all quasi-isomorphisms become isomorphisms.

5.34. We define derived category $D(\mathcal{A})$ for an abelian category $\mathcal{A}$ as the localization of the homotopy category $K(\mathcal{A})$ with respect to the class of all quasi-isomorphisms. A morphism between complexes $C_\bullet$ and $D_\bullet$ is no more only class of chain maps. Instead using universality, we have following roof diagram

$$C_\bullet \xleftarrow{f} F_\bullet \xrightarrow{g} D_\bullet \tag{14}$$

where f is a quasi-isomorphism and g is a morphism in $K(\mathcal{A})$. In a sense, it is more natural way to construct homotopy categories.

Derived categories are natural playground for derived functors, higher algebra, representation theory, and mathematical physics.

5. Mapping Cones and Triangulated Category

These sections are in progress (updated 05-Nov-25).

6. Homological Dimension

7. Derived Functors

8. More on Derived Categories