Old Superstring Formalism

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Discussion: After the times of dual resonance models [1–4] and establishment of bosonic string theory, the search for supersymmetric string theory began. The mentions of string actions that respected both fermionic and bosonic oscillators were 'out of the blue' (as Schwarz put it [5]). In a supersymmetric string theory, in addition to $X^\mu(\sigma,\tau)$, we have spinor degrees of freedom, Dirac spinors, and a 'zweibien' field, which is necessary[1]. These spinor degrees of freedom are described by D two-component spinors [7] which are

$$\lambda^{\mu A}(\sigma,\tau):\ \ A=1,2, \tag{1}$$

which transforms like both world-sheet and vector (in space time) but not necessarily as a spinor, for that we have another formalism which is more developed one [8, 9]. λ have anti-commutation properties though, it is a boson operator. Again, in a new formalism of superstring fermions anti-commutating operators are included. A problem arises for λ, as we now need two symmetries (local) to time components for both vectors, i.e $X^\mu(\sigma,\tau)$ and λ^μ. Using [10], it was solved with adding two local supersymmetry to reparametrization invariant action

$$S=-\frac{T}{2}\int d\sigma d\tau\ \eta_{\mu\nu}\sqrt{h}h^{\alpha\beta}\partial_\alpha X^\mu\ \partial_\beta X^\nu. \tag{2}$$

We have a Rarita-Schwinger field ψ^A_α [6, 11, 12], a field V^α_a related to metric $h_{\alpha\beta}$, and writing ρ^α two dimensional Dirac spinors, our action becomes

$$S=-\frac{T}{2}\int d\sigma d\tau\ \eta_{\mu\nu}V\left\{h^{\alpha\beta}\partial_\alpha X^\mu\ \partial_\beta X^\nu+iV^\alpha_a\bar{\lambda}^\mu\rho^a\partial_\alpha\lambda^\nu+2V^\alpha_aV^\beta_b\bar{\psi}_\alpha\rho^b\rho^a\lambda^\mu\left(\partial_\beta X^\nu+\frac{1}{2}\bar{\lambda}^\nu\psi_\beta\right)\right\}, \tag{3}$$

and this has the necessary supersymmetry requirement. In light-cone gauge, it can be written as

$$S_{l.c.g}=-\frac{T}{2}\int d\sigma d\tau\left(\partial^2_\alpha X^i+i\bar{\lambda}^i\rho^\alpha\partial_\alpha\lambda^i\right). \tag{4}$$

For (3) we will have the transverse modes in terms of X^i and λ^{Ai}. Before we write the mode expansions for X^i, we want to know what are the representation of ρ^α (Dirac spinors)

$$\rho^0=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\ \rho^1=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix} \tag{5}$$

and this will be our selected Majorana representation. We can now write equations of motion for λ

$$\left(\frac{\partial}{\partial\sigma}-\frac{\partial}{\partial\tau}\right)\lambda^{2i}=0 \tag{6}$$

$$\left(\frac{\partial}{\partial\sigma}+\frac{\partial}{\partial\tau}\right)\lambda^{i}=0,\quad A=1. \tag{7}$$

[1] As in supergravity [6].

We know the boundary conditions for X, boundary conditions for λ are as follows - derived from (6,7)

$$\lambda^i(\sigma = 0, \tau) = \lambda^{2i}(\sigma = 0, \tau) \tag{8}$$

$$\lambda^i(\pi, \tau) = \begin{cases} -\lambda^i(\pi, \tau) & \text{for bosons} \\ \lambda^i(\pi, \tau) & \text{for fermions} \end{cases} \tag{9}$$

we can see that for bosons, an extra minus sign is appearing whereas fermions doesn't. This can deduce that fermions should have normal mode expansions as X,

$$\lambda^{2i} = \sum_{h=-\infty}^{\infty} d_h^i \; e^{-ih(\tau-\sigma)} \tag{10}$$

$$\lambda^i = \sum_{h=-\infty}^{\infty} d_h^i \; e^{-ih(\tau+\sigma)} \tag{11}$$

$$[d_h^i, d_n^m] = \delta_{h+n,0} \; \delta^{im} \tag{12}$$

and for bosons,

$$\lambda^{2i} = \sum_{k=-\infty}^{\infty} b_k^i \; e^{-ik(\tau-\sigma)} \tag{13}$$

$$\lambda^i = \sum_{k=-\infty}^{\infty} b_k^i \; e^{-ik(\tau+\sigma)} \tag{14}$$

$$[b_k^i, b_o^m] = \delta_{k+o,0} \; \delta^{im}. \tag{15}$$

An important thing to note is because bosons have extra minus sign, k runs over half-integers $(\pm 1/2, \pm 3/2 \cdots)$ whereas j runs over all integers [5]. d and b are operators for fermions and bosons respectively. $\alpha'(mass)^2$ in the boson sector would be

$$\alpha'(\text{mass})^2 = \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i + \sum_{k=1/2}^{\infty} k b_{-k}^i b_k^i - c = M, \quad c = \frac{1}{2} \tag{16}$$

and the necessity of $c = 1/2$ is for Lorentz invariance. It comes from the ground state which is $b_{-1/2}^i \left|0\right\rangle$ and of course, ground state would be massless vector. We generally write Lorentz generators as

$$J^{ij} = l^{ij} - \mathrm{i} \sum_{n=1}^{\infty} \frac{1}{n} \left(\alpha_{-n}^i \alpha_n^j - \alpha_{-n}' \alpha_n^i \right) \tag{17}$$

where

$$l^{\mu\nu} = x^\mu p^\nu - x'' p^\mu \tag{18}$$

and we can change them to light-cone gauge. One important algebra in Lorentz consistencies (see [9]) is

$$\left[J^{i-}, J^{j-} \right] = 0. \tag{19}$$

We now write the Lorentz generators for bosons sector related to (16)

$$\begin{aligned} J^{ij} &= l^{ij} - \mathrm{i} \sum_{n=1}^{\infty} \frac{1}{n} \left(\alpha_{-n}^i \alpha_n^j - \alpha_{-n}^j \alpha_n^i \right) + K_0^{ij} \\ J^{i-} &= l^{i-} - \mathrm{i} \left(p^+\right)^{-1} \sum_{n=1}^{\infty} \frac{1}{n} \left(\alpha_{-n}^i \alpha_n^- - \alpha_{-n}^- \alpha_n^i \right) + \left(p^+\right)^{-1} \sum_{-\infty}^{\infty} K_{-n}^{ij} \alpha_n^j \end{aligned} \tag{20}$$

and we set

$$J^{+-} = l^{+-} \tag{21}$$

$$J^{i+} = l^{i+} \tag{22}$$

(we followed Schwarz notation from [5])

$$\begin{aligned} K_0^{ij} &= -\mathrm{i} \sum_{k=1/2}^{\infty} \left(b_{-k}^i b_k^j - b_{-k}^i b_k^i \right) \\ K_m^{ij} &= -\mathrm{i} \sum_{k=-\infty}^{\infty} b_{m-k}^i b_k^j, \quad m \neq 0 \end{aligned} \tag{23}$$

and

$$\alpha_n^- = \frac{1}{2} \sum_{m=-\infty}^{\infty} \alpha_{n-m}^i \alpha_m^i + \frac{1}{2} \sum_{k=-\infty}^{\infty} (k - n/2) b_{n-k}^i b_k^i, \quad n \neq 0 \tag{24}$$

$$\alpha_0^i \equiv p^i \tag{25}$$

We can similarly write the algebra for the fermionic sector by changing the oscillators b_k^i with d_h^i. With one difference being that c must set to $c = 0$ in (16) because we do not want negative norm masses squared in the fermionic sector, and then the ground state should be massless. More further, we see that (19) is only satisfied for this superstring in $D = 10$ [13], and it also implies the tracelessness of $T_{\mu\nu}$, hence ghosts free.

In (16) the number of boson states and in a modified fermionic version of (16) the number of fermion states are equal for $M = 0, 1, 2, \cdots$, that is supersymmetry . Nevertheless, that is only satisfied if the ground state is both Weyl (left-handed) and Majorana (real) spinor (and that provided). Moreover, in such (dual) models, the scattering of Fermion-Boson and fermion-fermion is the same. It is worth noticing that for D=4, the Weyl and Majorana requirement are equivalent. However, the condition for Weyl and Majorana's imposition on spinors in Minkowski space is cooperative in $D = 2$. A detailed discussion is carried in [14].

This is not a very good model. One important caveat is even λ anti-commutes, they are not Grassmann operators (and hence boson operators). A much more advanced covariant approach and spinor model (where no "spin-statistics" paradox has been given place) formalism should be (is already) adopted for supersymmetry in string theory [8, 15, 16].

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